Navier-Stokes equations in the whole space with an eddy viscosity

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Abstract

We study the Navier-Stokes equations with an extra eddy viscosity term in the whole space $\mathbb{R}^3$. We introduce a suitable regularized system for which we prove the existence of a regular solution defined for all time. We prove that when the regularizing parameter goes to zero, the solution of the regularized system converges to a turbulent solution of the initial system.

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In memory of Jean Leray

1 Introduction

Let us consider the incompressible Navier-Stokes Equations (NSE) in the whole space $\mathbb{R}^3$ with an extra eddy viscosity term:

$$(1.1) \qquad \begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} - \mathrm{div}(A(t,\mathbf{x})\nabla \mathbf{u}) + \nabla p = 0, & \text{(i)} \\ \mathrm{div}\, \mathbf{u} = 0, & \text{(ii)} \end{cases}$$

where $\mathbf{u} = \mathbf{u}(t,\mathbf{x}) = (u_1(t,\mathbf{x}), u_2(t,\mathbf{x}), u_3(t,\mathbf{x}))$ denotes the velocity, $p = p(t,\mathbf{x})$ the pressure, $t \geq 0$, $\mathbf{x} \in \mathbb{R}^3$, $\nu > 0$ is the kinetic viscosity and $A = A(t,\mathbf{x})$ is an eddy viscosity. In this paper, A is a scalar function.

This PDE system arises from turbulence modeling, the purpose of which is the calculation of averaged or filtered fields associated to a given turbulent flow. Eddy viscosities are usually introduced to model the Reynolds stress of such flows, according to the Boussinesq assumption (see for instance [11, 35]).

This system was already studied in the case of bounded domains with various boundary conditions (see in [11], chapters 6 to 8 for a comprehensive presentation). However, so far we know, it has never been investigated before in the case of the whole space, which motivates the present study.

We prove in this paper the existence of a turbulent solution to the NSE (1.1), global in time, through a suitable variational formulation on the basis of the assumptions:

i) $\mathbf{u}_0 \in L^2(\mathbb{R}^3)^3$, $\mathrm{div}\, \mathbf{u}_0 = 0$,

ii) $A \geq 0$, $A \in C_b(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3))$ and is of compact support in $\mathbb{R}^3$ uniformly in time.

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This main result is Theorem 2.3 in section 2, stated after giving necessary definitions. One of the key features of this solution is that it satisfies an energy inequality. The notion of turbulent solution was initially introduced by J. Leray [27] when $A = 0$, what makes our result a generalization of Leray's result. However, one part of Leray's program does not directly apply to the case $A \neq 0$, since the eddy viscosity term brings unexpected issues. Indeed, Leray was first looking for "strong" solutions of the NSE, that are shown to exist on a time interval $[0, T^\star[$ when $\mathbf{u}_0 \in L^\infty(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, in the case $A = 0$[1]. The argument is based on the Oseen's representation Theorem [33, 34] (see also section 3.3 below). The NSE in Leray's work are treated as Stokes equations, where the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u} = \mathrm{div}(\mathbf{u} \otimes \mathbf{u})$ is considered as a source term. This leads to an integral inequality satisfied by $||\mathbf{u}(t)||_\infty$ to get an estimate, which is one of the main building block of Leray's theory. We have first tried to generalize Oseen's theory to the equation

$$
\left\{ \begin{array}{ll} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} - \mathrm{div}(A(t, \mathbf{x}) \nabla \mathbf{u}) + \nabla p = \mathbf{f}, & \text{(i)} \\ \mathrm{div}\, \mathbf{u} = 0. & \text{(ii)} \end{array} \right. \tag{1.2}
$$

Unfortunately we have failed, which remains a difficult open problem. This is why we decided to treat the eddy diffusion term $-\mathrm{div}(A(t, \mathbf{x}) \nabla \mathbf{u})$ as a source term as well. The consequence is that terms depending on derivatives of $\mathbf{u}$ appear in the integral inequality satisfied by $||\mathbf{u}(t)||_\infty$, which can no longer yields an estimate. Therefore, we cannot directly pursue this strategy. In particular, we are not able to prove the existence of a strong solution to (1.1) on a time interval $[0, T^\star[$ when $A \neq 0$, whatever the choice of $\mathbf{u}_0$ and the assumptions satisfied by A. This is why things must be reconsidered, which motivates our developments that do not appear in Leray's paper. We provide in subsection 7.1 further technical details about this issue.

In this paper, the turbulent solution is constructed as a limit of regular solutions when $\varepsilon \to 0$ of the regularized NSE,

$$
\left\{ \begin{array}{l} \partial_t \mathbf{u} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} - \overline{\mathrm{div}\,(A \nabla \overline{\mathbf{u}})} + \nabla p = 0, \\ \mathrm{div}\, \mathbf{u} = 0, \\ \mathbf{u}_{t=0} = \overline{\mathbf{u}_0}, \end{array} \right. \tag{1.3}
$$

where $\overline{\psi} = \rho_\varepsilon \star \psi$ for a given mollifier ρ, and $\rho_\varepsilon = \varepsilon^{-3} \rho(\mathbf{x}/\varepsilon)$. The regularized convection term $\overline{\mathbf{u}} \cdot \nabla \mathbf{u}$ was initially introduced by J. Leray. He was able to show that the theory about strong solutions mentioned above, directly applies to the regularized system, and that one can take $T^\star = \infty$ in this case, which does not apply when $A \neq 0$.

The novelty is the introduction in section 3 of the regularized eddy viscosity term $-\overline{\mathrm{div}\,(A \nabla \overline{\mathbf{u}})}$, chosen in order to preserve the dissipation feature of the eddy viscosity. .

A large part of the paper is devoted to study the regularized NSE (1.3). We prove in section 3 that the Oseen representation formula applies to this system. However, the building block of this study is the section 4, where we show a series of *a priori* estimates satisfied by any regular solutions $(\mathbf{u}, p)$ of (1.3) and its derivatives. In particular, we derive from (1.3) estimates global in time for the L^∞ and L^2 norms of $D^m \mathbf{u}$, that is $||D^m \mathbf{u}(t)||_{0,\infty}$ and $||D^m \mathbf{u}(t)||_{0,2}$, in terms of the L^2 norm of the initial data and ε. This derivation is divided in four main steps.

◇ By the Oseen representation formula [33, 34], we show that $t \to ||D^m \mathbf{u}(t)||_{0,\infty}$ and $t \to ||D^m \mathbf{u}(t)||_{0,2}$ verify non linear integral inequalities,

[1] The big issue is to know if there is a solution such that $||\mathbf{u}(t)||_\infty \to \infty$ as $t \to T^\star$, $t < T^\star < \infty$...

- ◇ By generalizing to non linear cases principles based on Volterra's equations, which we have called the "V-maximum principle" (see section B.3 in appendix B), we get local in time estimates for $||D^m\mathbf{u}(t)||_{0,\infty}$ and $||D^m\mathbf{u}(t)||_{0,2}$ over a time interval $[0, t_\varepsilon^{(m)}(||\mathbf{u}_0||_{0,2})]$. In particular, we show that the application $x \to t_\varepsilon^{(m)}(x)$ is non increasing.
- ◇ We prove local in time estimates for derivatives of the pressure, $||D^m p(t)||_{0,\infty}$ and $||D^m p(t)||_{0,2}$, by the Calderón-Zygmund Theorem.
- ◇ We show that $\mathbf{u}$ satisfy the energy balance, which, combined with the monotonicity of $x \to t_\varepsilon^{(m)}(x)$, allows us to extend the local in time estimates to all $t \in [0, \infty[$.

This analysis, combined with a fixed point procedure, yields (section 5) the existence of a unique regular solution to System (1.3), global in time, which means a solution of class C^∞ in time and space defined for $t \in [0, \infty[$, with no singularity, the H^m norms of which are driven by the L^2 norm of $\mathbf{u}_0$, ε, the shape of ρ and its derivatives. This solution satisfies the energy balance, which provides valuable estimates that do not depend on ε.
We then show that the solution of the regularized NSE (1.3) converges to a turbulent solution of the NSE (1.1) when $\varepsilon \to 0$ in section 6. The proof makes use of an estimate of the solution of (1.3) for large values of $|\mathbf{x}|$, uniform in ε, which allows to apply standard compactness arguments on bounded domains. The assumption "A is of compact support uniformly in t" plays a role at this stage, and it is likely that it could be replaced by a suitable decay assumption of A for large values of $|\mathbf{x}|$.
We conclude the paper in section 7 by a series of remarks and additional open problems. We also make natural connections between the present work and models such as Bardina (see Layton-Lewandowski [24, 25], Cao-Lunasin-Titi [10]) and Leray-α (see Cheskidov-Holm-Olson-Titi [14]).
Finally, the appendice A aims to prove the basic estimates about the Oseen's tensor. The appendice B is devoted to the non linear Volterra equations and the V-maximum principle.

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2 Regular and turbulent solutions

2.1 Regular solutions

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$,

$$D^\alpha \mathbf{u} = (D^\alpha u_1, D^\alpha u_2, D^\alpha u_3), \quad D^\alpha u_i = \frac{\partial^{|\alpha|} u_i}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

For any given $m \in \mathbb{N}$, when we write $D^m \mathbf{u}$ we assume that $D^\alpha \mathbf{u}$ is well defined whatever α such that $|\alpha| = m$, and in practical calculations

$$|D^m \mathbf{u}| = \sup_{|\alpha|=m} |D^\alpha \mathbf{u}|.$$

The standard Sobolev space $W^{m,p}(\mathbb{R}^3)$ is equipped with the norm

$$||w||_{m,p} = \sum_{j=0}^{m} ||D^j w||_{L^p(\mathbf{R}^3)},$$

$H^m(\mathbb{R}^3) = W^{m,2}(\mathbb{R}^3)$. In this section, we assume temporarily that $\mathbf{u}_0 \in C^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and satisfies $\operatorname{div} \mathbf{u}_0 = 0$, and $A \in C(\mathbb{R}_+, C^1(\mathbb{R}))$.

Definition 2.1. *We say that $(\mathbf{u}, p)$ is a regular solution of the NSE (1.1) over the time interval $[0, T[$, if*

i) $\mathbf{u}, \partial_t \mathbf{u}, \nabla \mathbf{u}, D^2\mathbf{u}, p, \nabla p$ are well defined and continuous in t and x for $(t, \mathbf{x}) \in]0, T[\times \mathbb{R}^3$, and satisfy the relations (1.1.i) and (1.1.ii) in $\mathbb{R}^3$ at all $t \in]0, T[$,

ii) $\forall \tau < T$, $\mathbf{u} \in L^\infty([0, \tau], L^2(\mathbb{R}^3)^3) \cap L^\infty([0, \tau] \times \mathbb{R}^3)^3$,

iii) $(\mathbf{u}(t, \cdot))_{t>0}$ uniformly converges to $\mathbf{u}_0$ and in $H^1(\mathbb{R}^3)^3$ as $t \to 0$.

The pressure at any time t is solution of the elliptic equation

$$\Delta p = \operatorname{div}[\operatorname{div}(-\mathbf{u} \otimes \mathbf{u} + A\nabla \mathbf{u})],$$

which gives p once $\mathbf{u}$ is calculated. This is why the velocity $\mathbf{u}$ is sometimes referred to as the solution of the NSE instead of $(\mathbf{u}, p)$, for which we set:

$$W(t) = \int_{\mathbf{R}^3} |\mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x} = ||\mathbf{u}(t, \cdot)||_{0,2}^2, \tag{2.1}$$

$$J(t) = ||\nabla \mathbf{u}(t, \cdot)||_{0,2}, \tag{2.2}$$

$$V(t) = \sup_{\mathbf{x} \in \mathbf{R}^3} |\mathbf{u}(t, \mathbf{x})| = ||\mathbf{u}(t, \cdot)||_{0,\infty}, \tag{2.3}$$

$$V_m(t) = \sup_{\mathbf{x} \in \mathbf{R}^3} |D^m \mathbf{u}(t, \mathbf{x})| = ||D^m \mathbf{u}(t, \cdot)||_{0,\infty}. \tag{2.4}$$

At this stage, one of these quantities could be infinite at some date t. We also set

$$K_A(t) = \left(\int_{\mathbf{R}^3} A(t, \mathbf{x}) |\nabla \mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}} = ||\sqrt{A(t, \cdot)} \nabla \mathbf{u}(t, \cdot)||_{0,2}. \tag{2.5}$$

Definition 2.2. *We say that the solution becomes singular at T if $V(t) \to \infty$ when $t \to T$, $t < T$.*

We already know from J. Leray [27]:

Theorem 2.1. *Assume $A = 0$. Then there exists $T = O(\nu V^{-2}(0))$ such that the NSE (1.1) have a unique regular solution $(\mathbf{u}, p)$ over the time interval $[0, T[$, which satisfies in addition, $\mathbf{u} \in L^2([0, T], H^1(\mathbb{R}^3)^3) \cap C([0, T], L^2(\mathbb{R}^3)^3)$, and verifies the energy equality for any $t \in [0, T[$,*

$$\frac{1}{2} W(t) + \nu \int_0^t J^2(t') dt' = \frac{1}{2} W(0). \tag{2.6}$$

Finally, if $\nu^{-3} W(0) V(0)$ or $\nu^{-4} W(0) J^2(0)$ is small enough, the solution has no singularity and can be extended for all $t \in [0, \infty[$.

J. Leray also proved that the regular solution is of class C^∞ is space and time in the interval $]0,T[$, the quantities $||\mathbf{u}(t,\cdot)||_{m,2}$ and $V_m(t)$ being controled by $W(0)$, $V(0)$ and $J(0)$. Unfortunately, we are not able to generalize these results when $A \neq 0$ (see aditional comments in subsection 7.1).

Remark 2.1. *Since J. Leray, various definitions of regular solutions to the NSE when $A = 0$ and results of local strong solutions have been established by many different techniques. See for instance Fujita-Kato [17] and Kato [23], as well as Meyer-Cannone [9], Chemin [12] and Chemin-Gallagher [13] for further developements and references inside.*

2.2 Turbulent solutions

The notion of turbulent solution is based on a variational formulation and the energy inequality. The choice of the test vector fields space is essential. Within our framework, the space E_σ specified below seems natural:

$$\mathrm{E}_\sigma = \Big\{ \mathbf{w} \in L^1_{loc}(\mathbb{R}_+, H^3(\mathbb{R}^3)^3) \quad \text{s.t.} \quad \mathbf{w} \in C(\mathbb{R}^+, L^2(\mathbb{R}^3)^3), \tag{2.7}$$
$$\nabla \mathbf{w} \in L^\infty(\mathbb{R}, C_b(\mathbb{R}^3)^3), \quad \frac{\partial \mathbf{w}}{\partial t} \in L^1_{loc}(\mathbb{R}_+, L^2(\mathbb{R}^3)^3), \quad \operatorname{div} \mathbf{w} = 0 \Big\}.$$

This choice will be clear by the end of the paper. As usual, to find out the variational formulation, we take the dot product of the equation with a vector test field $\mathbf{w} \in \mathrm{E}_\sigma$ and we apply the Stokes formula, if the *a priori* solution $(\mathbf{u}, p)$ and its derivatives satisfy suitable integrability conditions, what we assume at this stage. The time derivative $\partial_t \mathbf{u}$ also adresses an issue. In our formulation, it is considered in the sense of the distribution. Then we formally get at a given time t:

$$\begin{cases} \displaystyle\int_{\mathbf{R}^3} \mathbf{u}_0(\mathbf{x}) \cdot \mathbf{w}(0,\mathbf{x}) d\mathbf{x} = \int_{\mathbf{R}^3} \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{w}(t,\mathbf{x}) d\mathbf{x} \\ \displaystyle + \int_0^t \int_{\mathbf{R}^3} [\mathbf{u}(t',\mathbf{x}) \otimes \mathbf{u}(t',\mathbf{x})] : \nabla \mathbf{w}(t',\mathbf{x}) d\mathbf{x} dt' \\ \displaystyle - \int_0^t \int_{\mathbf{R}^3} \mathbf{u}(t',\mathbf{x}) \cdot \left[\operatorname{div}((\nu + A(t',\mathbf{x}))\nabla \mathbf{w}(t',\mathbf{x})) + \frac{\partial \mathbf{w}(t',\mathbf{x})}{\partial t'} \right] d\mathbf{x} dt'. \end{cases} \tag{2.8}$$

Notice that as $(\nabla p, \mathbf{w}) = 0$ because $\operatorname{div} \mathbf{w} = 0$, the pressure is missing from this variational formulation, only involving the velocity $\mathbf{u}$, which is standard in NSE's framework.

Definition 2.3. *Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)^3$ such that $\operatorname{div} \mathbf{u}_0 = 0$. We say that $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$ is a turbulent solution to the NSE (1.1) with $\mathbf{u}_0$ as initial datum, if the following conditions are fulfilled:*

i) For all $t \geq 0$, $\mathbf{u}(t,\cdot) \in L^2(\mathbb{R}^3)^3$,

ii) $\mathbf{u} \in L^2(\mathbb{R}_+, H^1(\mathbb{R}^3)^3)$ and the following energy inequality holds:

$$\frac{1}{2} W(t) + \nu \int_0^t J^2(t') dt' + \int_0^t K_A(t') dt' \leq \frac{1}{2} W(0), \tag{2.9}$$

iii) For all $t \geq 0$, and for all $\mathbf{w} \in E_\sigma$, equality (2.8) holds.

It is easily checked that when $\mathbf{u}$ satisfies the items i) and ii) in Definition 2.3, then all the integrals in (2.8) are well defined for whatever $\mathbf{w} \in \mathrm{E}_\sigma$.

Assumption 2.1. *To avoid repetition, we will assume throughout the rest of the paper that* $\mathbf{u}_0 \in L^2(\mathbb{R}^3)^3$ *and* $\operatorname{div} \mathbf{u}_0 = 0$.

Theorem 2.2. *(J. Leray [27]) Assume* $A = 0$. *Then there exists a turbulent solution to the NSE (1.1). Moreover the turbulent solution becomes regular on the interval* $]CW(0)^2/\nu^5, \infty[$, *for some constant* C.

Remark 2.2. *Leray was considering a test vector field made of* C^∞ *vector field in space and time, which does not change much.*

The main result of this paper, which will be proved by the end of the paper, is the following:

Theorem 2.3. *Assume*

i) $A \geq 0$ *a.e in* $\mathbb{R}_+ \times \mathbb{R}^3$,

ii) $A \in C_b(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3))$,

iii) A *is with compact support in space uniformly in* t, *which means that there exists* $R_0 > 0$, *such that* $\forall\, t \geq 0$, $\forall\, \mathbf{x} \in \mathbb{R}^3$ *s.t.* $|\mathbf{x}| \geq R_0$, $A(t, \mathbf{x}) = 0$.

Then the NSE (1.1) have a turbulent solution.

In the statement above, C_b refers to as continuous bounded functions. The proof is based on regularizing the NSE by means of mollifiers sized by a parameter $\varepsilon > 0$, then taking the limit when $\varepsilon \to 0$.

Remark 2.3. *We do not know if the turbulent solution becomes regular after a given time* T *when* $A \neq 0$ *(see section 7.1 for additional comments).*

Remark 2.4. *Solutions based on a variational formulation like (2.8) are sometimes referred to as "very weak solutions" (see Lions-Masmoudi [30]).*

3 Regularized system

3.1 Mollifier

Let $\rho \in C^\infty(\mathbb{R}^3)$ denotes a non negative function with compact support such that

$$\int_{\mathbf{R}^3} \rho(\mathbf{x}) d\mathbf{x} = 1,$$

and let

$$\rho_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^3} \rho\left(\frac{\mathbf{x}}{\varepsilon}\right).$$

Any $U \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ being given, we set

$$\overline{U}(\mathbf{x}) = \rho_\varepsilon \star U(\mathbf{x}) = \int_{\mathbf{R}^3} \rho_\varepsilon\,(\mathbf{x} - \mathbf{y})\, U(\mathbf{y}) d\mathbf{y}.$$

It is well known that $\overline{U}$ is of class C^∞ and when $U \in L^p(\mathbb{R}^3)$, $1 \le p < \infty$, then $\overline{U}$ converges to U in $L^p(\mathbb{R}^3)$ when $\varepsilon \to 0$. We will need the following formal estimates:

$$||\overline{U}||_{0,\infty} \le ||U||_{0,\infty}, \tag{3.1}$$

$$||D^m\overline{U}||_{0,\infty} \le \frac{C_m}{\varepsilon^{3/2+m}}||U||_{0,2}, \tag{3.2}$$

$$||\overline{U} - U||_{m,2} \le C_m\, \varepsilon\, ||U||_{m-1,2}, \tag{3.3}$$

where C_m is a constant that only depends on m, the shape of ρ and its derivatives. These estimates as well as many others properties about regularization by convolution can be found for instance in [7, 27, 31].

Finally, we assume that the kernel ρ is an even function, so that the regularization operator $U \to \overline{U}$ is self adjoint in L^2.

3.2 Approximated system

We regularize the convection and the eddy viscosity terms as follows:

i) Following J. Leray, the convective term $\mathbf{u} \cdot \nabla \mathbf{u}$ is approximated by $\overline{\mathbf{u}} \cdot \nabla \mathbf{u}$,

ii) The eddy viscosity term $-\mathrm{div}(A\nabla\mathbf{u})$ is approximated by $-\overline{\mathrm{div}\,(A\nabla\overline{\mathbf{u}})}$.

This way of regularizing the eddy viscosity term provides the advantage that it preserves its dissipative feature. Indeed, we formally have by the self adjointness of the bar operator:

$$(-\overline{\mathrm{div}\,(A\nabla\overline{\mathbf{u}})}, \mathbf{u}) = (-\mathrm{div}\,(A\nabla\overline{\mathbf{u}}), \overline{\mathbf{u}}) = (A\nabla\overline{\mathbf{u}}, \nabla\overline{\mathbf{u}}) = \int_{\mathbf{R}^3} A|\nabla\overline{\mathbf{u}}|^2 \ge 0, \tag{3.4}$$

as $A \ge 0$, and where $(\cdot,\cdot)$ denotes the scalar product in L^2.

According to Assumption 2.1, the initial datum also needs to be regularized. Thus, we recall what is the regularized NSE, already written in the introduction:

$$\begin{cases} \partial_t \mathbf{u} + \overline{\mathbf{u}} \cdot \nabla\mathbf{u} - \nu\Delta\mathbf{u} - \mathrm{div}\,(\overline{A\nabla\overline{\mathbf{u}}}) + \nabla p = 0, & \text{(i)} \\ \mathrm{div}\,\mathbf{u} = 0, & \text{(ii)} \\ \mathbf{u}_{t=0} = \overline{\mathbf{u}_0}. & \text{(iii)} \end{cases} \tag{3.5}$$

We adopt for the regularized NSE (3.5), the notion of regular solution given by Definition 2.1. By the end of the next section, we will have proved:

Theorem 3.1. *Assume $A \ge 0$, $A \in C_b(\mathbb{R}_+, L^\infty(\mathbb{R}^3)^3)$. Then the regularized NSE (3.5) have a unique regular solution $(\mathbf{u}, p) \in C(\mathbb{R}_+, H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3))$, $\forall\, m \in \mathbb{N}$, which satisfies the energy balance*

$$\frac{1}{2}W(t) + \nu\int_0^t J^2(t')dt' + \int_0^t K^2_{A,\varepsilon}(t')dt' = \frac{1}{2}W_\varepsilon(0). \tag{3.6}$$

Recall that $W(t) = ||\mathbf{u}(t,\cdot)||^2_{0,2}$ and $J(t) = ||\nabla\mathbf{u}(t,\cdot)||_{0,2}$ were initially defined by (2.1) and (2.2). The quantity $W_\varepsilon(0) = ||\overline{\mathbf{u}_0}||^2_{0,2}$ verifies

$$W_\varepsilon(0) \le W(0) = \int_{\mathbf{R}^3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x}. \tag{3.7}$$

We also have set

$$K_{A,\varepsilon}(t) = \left(\int_{\mathbf{R}^3} A(t,\mathbf{x}) |\nabla \overline{\mathbf{u}}(t,\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}}. \tag{3.8}$$

A similar result is in Leray's paper [27] when $A = 0$, section 26. His argument, based on the control of $V(t) = ||\mathbf{u}(t,\cdot)||_{0,\infty}$, does not work when $A \neq 0$ (the main reason is detailed in section 7.1). This is why we had to write an original proof of Theorem 3.1 when $A \neq 0$, based this time on the control of the H^m norms of the velocity, *i.e.* $||\mathbf{u}(t,\cdot)||_{m,2}$, for any $m \geq 0$. To do so, we will find out sharp estimates, the control parameters of which are the L^2 norm of $\mathbf{u}_0$ and ε. This led us to make improvements in the understanding of the equivalence between the equations and the integral representation, as well as in the processing of the pressure by modern regularity results, Sobolev spaces and the Calderón-Zygmund Theorem [38], which was not known as J. Leray was writing his paper.

Remark 3.1. *The assumption "A is with compact support in space uniformly in time" is not needed in this statement. Note that no further information about its gradient is required at this stage.*

Assumption 3.1. *Throughout the rest of the paper, we will assume at least that* $A \geq 0$, $A \in C_b(\mathbb{R}_+, L^\infty(\mathbb{R}^3))$.

3.3 Oseen representation

The proof of Theorem 3.1 is based on an integral formulation of the regularized NSE (3.5) by a suitable Kernel known as the Oseen's potential, recalled in this subsection.

Let us consider the evolutionary Stokes problem with a continuous source term $\mathbf{f}$ and a continuous initial datum $\mathbf{v}_0$:

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}_{t=0} = \mathbf{v}_0. \end{cases} \tag{3.9}$$

C. Oseen [33, 34] shown that there exists a tensor $\mathbf{T} = (T_{ij})_{1 \leq i,j \leq 3}$ such that if $(\mathbf{u}, p)$ is a regular solution of (3.9), then the velocity $\mathbf{u}$ solution of (3.9) verifies

$$\mathbf{u}(t,\mathbf{x}) = (Q \star \mathbf{v}_0)(t,\mathbf{x}) + \int_0^t \int_{\mathbf{R}^3} \mathbf{T}(t-t', \mathbf{x}-\mathbf{y}) \cdot \mathbf{f}(t',\mathbf{y})\, d\mathbf{y}, \tag{3.10}$$

where

$$Q(t,\mathbf{x}) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{|\mathbf{x}|^2}{4\nu t}}, \tag{3.11}$$

is the heat kernel and

$$(Q \star \mathbf{v}_0)(t,\mathbf{x}) = \int_{\mathbf{R}^3} Q(t, \mathbf{x}-\mathbf{y}) \mathbf{v}_0(\mathbf{y}) d\mathbf{y}.$$

The components of $\mathbf{T}$ can be specified as follows. Let

$$G(t,x) = \frac{1}{|x|} \int_0^{|x|} \frac{e^{-\frac{\rho^2}{4\nu t}}}{\sqrt{t}} \mathrm{d}\rho.$$

The function G satisfies the PDE

$$\Delta\left(\partial_t G + \nu\Delta G\right) = 0, \tag{3.12}$$

and the Oseen's tensor is given by

$$\forall\, i \neq j \neq k, \quad T_{ii} = -\frac{\partial^2 G}{\partial x_j^2} - \frac{\partial^2 G}{\partial x_k^2}, \quad \forall\, i \neq j, \quad T_{ij} = \frac{\partial^2 G}{\partial x_j \partial x_i}. \tag{3.13}$$

This tensor satisfies the inverse Euler system, where $L_i = (T_{i1}, T_{i2}, T_{i3})$,

$$\begin{cases} \partial_t L_i + \nu\Delta L_i - \nabla \dfrac{\partial}{\partial x_i}\left(\partial_t G + \nu\Delta G\right) = 0, \\ \operatorname{div} L_i = 0. \end{cases} \tag{3.14}$$

In Appendice A, the following estimates are proved:

$$|\mathbf{T}(t,\mathbf{x})| \leq \frac{C}{(|\mathbf{x}|^2 + \nu t)^{\frac{3}{2}}}, \tag{3.15}$$

$$\forall\, m \geq 0, \quad |D^m \mathbf{T}(t,\mathbf{x})| \leq \frac{C_m}{(|\mathbf{x}|^2 + \nu t)^{\frac{m+3}{2}}}, \tag{3.16}$$

C and C_m being some constants. We start with the following regularity result.

Lemma 3.1. *For all $T > 0$, $m \geq 0$, $D^m \mathbf{T} \in L^p([0,T], L^q(\mathbb{R}^3)$, for exponents (p,q) such that $q > 3/(m+3)$ and $p < (3/2)q'$, where $1/q + 1/q' = 1$.*

Proof. By the estimate (3.16) we get

$$\int_{\mathbf{R}^3} |D^m \mathbf{T}(t,\mathbf{x})|^q d\mathbf{x} \leq C \int_0^\infty \frac{r^2 dr}{(r^2 + \nu t)^{\frac{q(m+3)}{2}}} = \frac{1}{(\nu t)^{\frac{q(m+3)-3}{2}}} \int_0^\infty \frac{\rho^2 d\rho}{(\rho^2 + 1)^{\frac{q(m+3)}{2}}}. \tag{3.17}$$

Therefore, $D^m \mathbf{T} \in L^p([0,T], L^q(\mathbb{R}^3)$ for (p,q) such that

$$q(m+3) - 2 > 1, \quad \frac{p(q(m+3)-3)}{2q} < 1,$$

hence the result. □

In particular, for $m = 1$, we have the following corollary:

Corollary 3.1. *Let $t > 0$. Then $t' \to ||\nabla\, \mathbf{T}(t-t', \cdot)||_{0,1} \in L^1([0,t])$ and,*

$$||\nabla\, \mathbf{T}(t-t', \cdot)||_{0,1} \leq \frac{C}{\sqrt{\nu(t-t')}}. \tag{3.18}$$

Before stating the next result, we must specify some notations. Let $V = (V_{ij})_{1 \leq i,j \leq 3}$, $W = (W_{ij})_{1 \leq i,j \leq 3}$ two tensors, $V \cdot W = (V_{ij} W_{jk})_{1 \leq i,k \leq 3}$ their product . The vector $\operatorname{div} V$ is given component by component by:

$$[\operatorname{div} V]_i = \partial_j V_{ij}, \tag{3.19}$$

where $\partial_j = \partial/\partial x_j$. The vector $\nabla V : W$ is given by

$$[\nabla V : W]_i = \partial_k V_{ij} W_{jk}. \tag{3.20}$$

Finally

$$\operatorname{div}^2 V = \partial_i \partial_j V_{ij}. \tag{3.21}$$

Lemma 3.2. *Let* $(\mathbf{u}, p)$ *be a regular solution of the regularized NSE (3.5) over the time interval* $[0, T[$ *for a given* $T > 0$*. Then the velocity* $\mathbf{u}$ *satisfies for all* $t \in [0, T[$*,*

$$
(3.22) \qquad \begin{cases} \mathbf{u}(t,\mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t,\mathbf{x}) \\ \\ + \displaystyle\int_0^t \int_{\mathbf{R}^3} \nabla \mathbf{T}(t-t', \mathbf{x}-\mathbf{y}) : [\overline{\mathbf{u}}(t',\mathbf{y}) \otimes \mathbf{u}(t',\mathbf{y}) - \overline{A\nabla\overline{\mathbf{u}}}(t',\mathbf{y})] \, d\mathbf{y} dt', \end{cases}
$$

and the pressure p *is deduced from the velocity* $\mathbf{u}$ *by the formula:*

$$
(3.23) \qquad p(t,\mathbf{x}) = \frac{1}{4\pi} \mathrm{div}^2 \int_{\mathbf{R}^3} \frac{\overline{\mathbf{u}}(t,\mathbf{y}) \otimes \mathbf{u}(t,\mathbf{y}) - \overline{A\nabla\overline{\mathbf{u}}}(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y},
$$

Proof. Let

$$
(3.24) \qquad \mathbf{X}(t,x) = \overline{\mathbf{u}}(t,\mathbf{x}) \otimes \mathbf{u}(t,\mathbf{x}) - \overline{A\nabla\overline{\mathbf{u}}}(t,\mathbf{x}),
$$

so that the regularized NSE (3.5) can be written in the form

$$
(3.25) \qquad \begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = -\mathrm{div} \mathbf{X}, \\ \mathrm{div}\, \mathbf{u} = 0, \\ \mathbf{u}_{t=0} = \overline{\mathbf{u}_0}. \end{cases}
$$

The proof is divided in two steps:

Step 1) We first study the regularity of $\mathbf{X}$ in order to obtain the formula (3.23) and the formula: $\forall\, t \in [0, t[$,

$$
(3.26) \qquad \mathbf{u}(t,\mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t,\mathbf{x}) + \mathrm{div} \int_0^t \int_{\mathbf{R}^3} \mathbf{T}(t-t', \mathbf{x}-\mathbf{y}) \cdot \mathbf{X}(t',\mathbf{y}) dt'.
$$

Step 2) We prove that we can switch the integral and the derivative in the formula (3.26).

Step 1) On one hand we have

$$
||\overline{\mathbf{u}}(t',\cdot) \otimes \mathbf{u}(t',\cdot)||_{0,2} \leq ||\overline{\mathbf{u}}(t,\cdot)||_{0,\infty} ||\mathbf{u}(t,\cdot)||_{0,2},
$$

and by (3.2),

$$
||\overline{\mathbf{u}}(t,\cdot)||_{0,\infty} \leq C\varepsilon^{-\frac{3}{2}} ||\mathbf{u}(t,\cdot)||_{0,2},
$$

which leads to

$$
(3.27) \qquad ||\overline{\mathbf{u}}(t',\cdot) \otimes \mathbf{u}(t',\cdot)||_{0,2} \leq C\varepsilon^{-\frac{3}{2}} W(t).
$$

On an other hand, similar calculus inequalities lead to

$$
(3.28) \qquad ||\overline{A\nabla\overline{\mathbf{u}}}(t',\cdot)||_{0,2} \leq C ||N_A||_\infty \varepsilon^{-1} \sqrt{W(t)},
$$

where we have set

$$
(3.29) \qquad N_A(t) = ||A(t,\cdot)||_{0,\infty}.
$$

Therefore,

$$
(3.30) \qquad ||\mathbf{X}(t,\cdot)||_{0,2} \leq C\varepsilon^{-1} \left[\varepsilon^{-\frac{1}{2}} W(t) + ||N_A||_\infty \sqrt{W(t)} \right].
$$

Then according to the item ii) in the definition (2.1),

$$\forall \tau \in [0,T[, \quad \mathbf{X} \in L^\infty([0,\tau], L^2(\mathbb{R}^3)^9). \tag{3.31}$$

By (3.2) combined with (4.10), we obtain

$$||\overline{A(t',\cdot)\nabla\bar{\mathbf{u}}(t,\cdot)}||_{0,\infty} \le C\varepsilon^{-5/2}||N_A||_\infty\sqrt{W(t)}, \tag{3.32}$$

and by (3.1), we finally have

$$||\mathbf{X}(t,\cdot)||_{0,\infty} \le V(t)^2 + C\varepsilon^{-5/2}||N_A||_\infty\sqrt{W(t)}. \tag{3.33}$$

Then, again by the item ii) in the definition (2.1), for any $\tau < T$,

$$\forall \tau \in [0,T[, \quad \mathbf{X} \in L^\infty([0,\tau]\times\mathbb{R}^3)^9. \tag{3.34}$$

Moreover, $D^\alpha\mathbf{X}$ is continuous whatever $|\alpha| = 2$, in view of item i) of Definition 2.1 and the regularizing effect of the bar operator. Therefore, (3.31) and (3.34) being satisfied, we can apply the lemma 8 in [27] and we get (3.23) as well as (3.26).

Step 2)[2] In what follows we set

$$N_{\tau,\mathbf{X}} = ||\mathbf{X}||_{L^\infty([0,\tau]\times\mathbf{R}^3)}. \tag{3.35}$$

Let

$$V(t,\mathbf{x}) = \int_0^t\int_{\mathbf{R}^3}\mathbf{T}(t-t',\mathbf{x}-\mathbf{y})\cdot\mathbf{X}(t',\mathbf{y})dt'. \tag{3.36}$$

Let $\mathbf{h}_i = h\mathbf{e}_i$, for $i = 1,2,3$. Let $V_h(t,\mathbf{x})$ denotes the function

$$V_h(t,\mathbf{x}) = \frac{1}{h}\left[V(t,\mathbf{x}+\mathbf{h}_i) - V(t,\mathbf{x})\right].$$

on one hand we have,

$$\partial_i V(t,\mathbf{x}) = \lim_{h\to 0} V_h(t,\mathbf{x}), \tag{3.37}$$

and on the other hand,

$$V_h(t,\mathbf{x}) = \int_0^t\int_{\mathbf{R}^3}\frac{1}{h}[\mathbf{T}(t-t',\mathbf{x}-\mathbf{y}+\mathbf{h}_i) - \mathbf{T}(t-t',\mathbf{x}-\mathbf{y})] : \mathbf{X}(t',\mathbf{y})dt'. \tag{3.38}$$

Let $U_h(t,t';\mathbf{x},\mathbf{y})$ denotes the function

$$U_h(t,t';\mathbf{x},\mathbf{y}) = \frac{1}{h}[\mathbf{T}(t-t',\mathbf{x}-\mathbf{y}+\mathbf{h}_i) - \mathbf{T}(t-t',\mathbf{x}-\mathbf{y})] : \mathbf{X}(t',\mathbf{y}),$$

so that

$$V_h(t,\mathbf{x}) = \int_0^t\int_{\mathbf{R}^3} U_h(t,t';\mathbf{x},\mathbf{y})d\mathbf{y}dt'.$$

[2] Leray states his Lemma 8 as a consequence of a uniqueness result. If the uniqueness result is entirely proved, there is in his paper [27] no proof of this lemma 8, although it is quite reasonnable. In this present step 2), we are kniting things backward and we indirectly more or less prove this lemma 8, without considering the uniqueness argument, based on the L^2 integrabilities of $\mathbf{X}$ and $\mathbf{u}$, which holds in our case. This proof mainly explains the underlying machine for the derivation of the following H^m estimates.

We will pass to the limit in this integral for $h \to 0$, by two consecutive applications of the Lebesgue's theorem. By definition, for any given $0 \le t' \le t$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$,

$$\lim_{h \to 0} U_h(t, t'; \mathbf{x}, \mathbf{y}) = \partial_i \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : \mathbf{X}(t', \mathbf{y}). \tag{3.39}$$

We write the standard formula:

$$\frac{1}{h}[\mathbf{T}(t - t', \mathbf{x} - \mathbf{y} + \mathbf{h}_i) - \mathbf{T}(t - t', \mathbf{x} - \mathbf{y})] = \int_0^1 \partial_i \mathbf{T}(t - t', \mathbf{x} - \mathbf{y} + s\mathbf{h}_i) ds. \tag{3.40}$$

By consequence, by (3.16), we have for any fixed $0 \le t' < t$, $\mathbf{x} \in \mathbb{R}^3$, and any h such that $|h| \le 1/2$,

$$|U_h(t, t'; \mathbf{x}, \mathbf{y})| \le C N_{t, \mathbf{x}} \left[\frac{\mathbb{I}_{B(\mathbf{x}, 1)}}{\nu^2 (t - t')^2} + \frac{\mathbb{I}_{B(\mathbf{x}, 1)^c}}{(|\mathbf{x} - \mathbf{y}|^2/4 + \nu(t - t'))^2} \right] = H(t, t'; \mathbf{x}, \mathbf{y}),$$

and we observe that $H(t, t'; \mathbf{x}, \mathbf{y}) \in L^1_{\mathbf{y}}(\mathbb{R}^3)$ for any given $(t, t'; \mathbf{x})$. Then by Lebesgue's Theorem,

$$\lim_{h \to 0} \int_{\mathbf{R}^3} U_h(t, t'; \mathbf{x}, \mathbf{y}) d\mathbf{y} = \int_{\mathbf{R}^3} \partial_i \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : \mathbf{X}(t', \mathbf{y}) d\mathbf{y} = v(t, t'; \mathbf{x}), \tag{3.41}$$

for all $0 \le t' < t$, $\mathbf{x} \in \mathbb{R}^3$. Let

$$v_h(t, t'; \mathbf{x}) = \int_{\mathbf{R}^3} U_h(t, t'; \mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

In other words, we have proved that for any fixed $(t, \mathbf{x}) \in]0, T[\times \mathbb{R}^3$,

$$\forall\, t' \in [0, t[, \quad \lim_{h \to 0} v_h(t, t'; \mathbf{x}) = v(t, t'; \mathbf{x}).$$

Notice that

$$\int_0^t v_h(t, t'; \mathbf{x}) dt' = V_h(t, \mathbf{x}),$$

so that we must take the limit in the integral above when $h \to 0$. By using (3.40) combined with (3.16) once again, we obtain by Fubini's Theorem,

$$|v_h(t, t'; \mathbf{x})| \le C N_{t, \mathbf{x}} \int_0^1 ds \int_{\mathbf{R}^3} \frac{d\mathbf{y}}{(|\mathbf{x} - \mathbf{y} + s\mathbf{h}_i|^2 + \nu(t - t'))^2}, \tag{3.42}$$

which leads to, by the same calculation as that in the proof of Lemma 3.1,

$$|v_h(t, t'; \mathbf{x})| \le \frac{C N_{t, \mathbf{x}}}{\sqrt{\nu(t - t')}} \in L^1([0, t]). \tag{3.43}$$

Then, by Lebesgue's Theorem once again,

$$\lim_{h \to 0} \int_0^t v_h(t, t'; \mathbf{x}) dt' = \lim_{h \to 0} V_h(t, \mathbf{x}) = \int_0^t v(t, t'; \mathbf{x}) dt', \tag{3.44}$$

which means by (3.37) and (3.41),

$$\partial_i \int_0^t \int_{\mathbf{R}^3} \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) \cdot \mathbf{X}(t', \mathbf{y}) dt' = \int_0^t \int_{\mathbf{R}^3} \partial_i \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) \cdot \mathbf{X}(t', \mathbf{y}) dt',$$

hence formula (3.22) by (3.26). □

Remark 3.2. *By a similar reasonning based on Lebesgue's Theorem, we also can prove that* $\mathbf{u} \in C([0, T[, L^2(\mathbb{R}^3)^3) \cap C([0, T[, L^\infty(\mathbb{R}^3)^3)$. *More generally, the time continuity of the velocity in* H^m *will be proved in Lemma 4.6 below by directly using the equation.*

4 A priori estimates and energy balance

We derive from the integral representation (3.22) *a priori* H^m and $W^{m,\infty}$ estimates satisfied by the velocity part of any regular solution of the regularized NSE (1.3). We start with local in time estimates which we afterwards extend for all time. We also focus on the obtention of energy equalities, which requires estimates for the pressure.
The constants involved in the inequalities of this section, depend on those involved in the inequality (3.16), on the shape of ρ and its derivatives, which will be not systematically mentioned.
Throughout this section 4, $(\mathbf{u}, p)$ denotes a regular solution of the regularized NSE (3.5). Moreover, Assumption 2.1 and Assumption 3.1 hold.

4.1 Local time L^2 and L^∞ estimates

We recall that the definition of a regular solution requires that the velocity satisfies $\forall \tau < T$, $\mathbf{u} \in L^\infty([0,\tau], L^2(\mathbb{R}^3)^3) \cap L^\infty([0,\tau] \times \mathbb{R}^3)^3$. As we saw it in the previous section, this information led to the integral representation formula (3.22). The aim of this section is to show that the $L^\infty(L^2)$ and $L^\infty(L^\infty)$ norms of $\mathbf{u}$ are entirely driven by the initial kinetic energy, namely the quantity $W(0)$, which will be derived from this integral representation formula (3.22) combined with the V-maximum principle set out in Appendice B.

Lemma 4.1. *There exists $t_\varepsilon(W(0)) > 0$ such that $\forall\, t \in [0, t_\varepsilon(W(0))]$,*

$$W(t) \leq 4W(0). \tag{4.1}$$

$$V(t) \leq 2C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}. \tag{4.2}$$

Moreover the function $x \to t_\varepsilon(x)$ is a non increasing function of x.

Proof. We start by proving (4.1) over a time interval $[0, t_{1,\varepsilon}(W(0))]$. The field $\mathbf{u}$ is a regular solution to the regularized NSE, therefore as we already have said, it belongs to $L^\infty([0,T/2], L^2(\mathbb{R}^3)^3)$. Let

$$W_{T/2} = \sup_{t\in[0,T/2]} W(t) < \infty.$$

Assume that

$$4W(0) < W_{T/2}, \tag{4.3}$$

otherwise take $t_{1,\varepsilon}(W(0)) = T/2$. Starting from this and working on the time interval $[0, T/2]$, we deduce from the integral representation (3.22) combined with the Young's inequality,

$$\sqrt{W(t)} \leq \sqrt{W(0)} + \int_0^t ||\nabla \mathbf{T}(t-t', \cdot)||_{0,1} ||\mathbf{X}(t', \cdot)||_{0,2} dt', \tag{4.4}$$

where $\mathbf{X}$ is defined by (3.24). Therefore, by (3.18),

$$\sqrt{W(t)} \leq \sqrt{W(0)} + C\int_0^t \frac{||\mathbf{X}(t', \cdot)||_{0,2}}{\sqrt{\nu(t-t')}} dt'. \tag{4.5}$$

The estimate (3.30) yields

$$\sqrt{W(t)} \leq \sqrt{W(0)} + C\varepsilon^{-1}\int_0^t \frac{\varepsilon^{-1/2}W(t') + ||N_A||_\infty\sqrt{W(t')}}{\sqrt{\nu(t-t')}} dt'. \tag{4.6}$$

Let $P(f)$ be defined by

$$(4.7) \qquad \begin{cases} P(f) = 0 & \text{if } f \le 0, \\ P(f) = C\varepsilon^{-1}[\varepsilon^{-1/2} f^2 + ||N_A||_\infty f] & \text{if } 0 \le f \le \sqrt{W_{T/2}}, \\ P(f) = C\varepsilon^{-1}[\varepsilon^{-1/2} W_{T/2} + ||N_A||_\infty \sqrt{W_{T/2}}] & \text{if } f \le \sqrt{W_{T/2}}. \end{cases}$$

As $t \in [0, T/2]$ and $W(t) \le W_{T/2}$, the inequality (4.6) shows that $t \to \sqrt{W(t)}$ is a subsolution of the non linear Volterra equation (see Appendix B)

$$(4.8) \qquad f(t) = \sqrt{W(0)} + \int_0^t \frac{P(f)(t')}{\sqrt{\nu(t-t')}} dt',$$

with the kernel

$$K(t) = \frac{1}{\sqrt{\nu t}} \in L^1([0,T]).$$

In this equation, P is indeed a non decreasing Lipchitz continuous function. As $4W(0) < W_{T/2}$, the constant function $g(t) = 2\sqrt{W(0)}$ is a supersolution of Equation (4.8) over the time interval $[0, t_\varepsilon(W(0))]$, where

$$(4.9) \qquad t_{1,\varepsilon}(x) = \inf\left(\frac{\nu\varepsilon^2}{4C^2(||N_A||_\infty + \varepsilon^{-1/2}\sqrt{x})^2}, \frac{T}{2} \right).$$

We then deduce from the V-maximum principle proved in Lemma B.4, that

$$(4.10) \qquad \forall t \in [0, t_{1,\varepsilon}(W(0))], \quad \sqrt{W(t)} \le 2\sqrt{W(0)}.$$

Notice that the function $x \to t_{1,\varepsilon}(x)$ given by (4.9) is non increasing.

Let us now prove (4.2). Take $t, t' \in [0, t_{1,\varepsilon}(W(0))]$. Combining (3.32) with (4.10), we obtain

$$(4.11) \qquad ||\overline{A(t',\cdot)\nabla\overline{\mathbf{u}}(t',\cdot)}||_{0,\infty} \le C\varepsilon^{-5/2}||N_A||_\infty\sqrt{W(0)}.$$

Moreover, repeating the combination of (3.2) with (4.1) gives

$$||\overline{\mathbf{u}}(t,\cdot)||_{0,\infty} \le C\varepsilon^{-\frac{3}{2}}||\mathbf{u}(t,\cdot)||_{0,2} \le C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)},$$

hence

$$(4.12) \qquad ||\overline{\mathbf{u}}(t',\cdot) \otimes \mathbf{u}(t',\cdot)||_{0,\infty} \le C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}V(t'),$$

which improves the first estimate (3.33) of $||\mathbf{X}(t',\cdot)||_{0,\infty}$ by giving

$$(4.13) \qquad ||\mathbf{X}(t',\cdot)||_{0,\infty} \le C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}\left[V(t') + \varepsilon^{-1}||N_A||_\infty\right].$$

Finally, as

$$(4.14) \qquad V_\varepsilon(0) = ||\overline{\mathbf{u}_0}||_{0,\infty} \le C\varepsilon^{-\frac{3}{2}}||\mathbf{u}||_{0,2} = C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)},$$

we get

$$(4.15) \qquad ||Q \star \overline{\mathbf{u}_0}(t,\mathbf{x})||_{0,\infty} \le V_\varepsilon(0) \le C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}.$$

We combine (4.13) and (4.15) with the formula (3.22) and (3.18), which yields for any $t \in [0, t_{1,\varepsilon}]$,

$$V(t) \leq C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}\left(1 + \int_0^t \frac{V(t') + \varepsilon^{-1}||N_A||_\infty}{\sqrt{\nu(t-t')}} dt'\right). \tag{4.16}$$

Therefore, using the V-maximum principle again gives (we skip the details)

$$\forall\, t \in [0, t_\varepsilon(W(0))], \quad V(t) \leq 2C\varepsilon^{-\frac{3}{2}}\sqrt{W(0)},$$

where, after a straighforward calculation,

$$t_\varepsilon(x) = \inf\left(\frac{2\varepsilon\nu}{(\varepsilon^{-\frac{1}{2}}||N_A||_\infty + 4C\sqrt{x})^2}, t_{1,\varepsilon}\right).$$

The function $x \to t_\varepsilon(x)$ is indeed non increasing. □

4.2 Local time H^m and $W^{1,\infty}$ estimates

To get H^m estimates, uniform in time, we will argue by induction in taking consecutive derivatives of the integral formula (3.22), which comes back to the issue of switching integrals and derivatives. The first lemma of this section is the basis to justify the first switching, then initializing the induction.

Lemma 4.2. *Recall that $V_1(t) = ||D\mathbf{u}(t,\cdot)||_{0,\infty}$. Then, $\forall\, t \in [0, t_\varepsilon(W(0))]$,*

$$V_1(t) \leq C\nu^{-\frac{3}{2}}\varepsilon^{-\frac{5}{3}}\sqrt{W(0)}\left[\varepsilon^{-\frac{1}{3}}\sqrt{W(0)} + 1\right]\sqrt{t} + C'\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}. \tag{4.17}$$

Proof. We can apply to this case the inequality (2.14) page 213 in Leray [27], which yields: $\forall\, t \in [0, t_\varepsilon(W(0))]$, $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$,

$$|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})| \leq C|\mathbf{x}-\mathbf{y}|^{\frac{1}{2}} \int_0^t \frac{||(\overline{\mathbf{u}} \otimes \mathbf{u})(t',\cdot)||_{0,\infty} + ||\overline{A\nabla\overline{\mathbf{u}}}(t',\cdot)||_{0,\infty}}{[\nu(t-t')]^{\frac{3}{4}}} dt' + |Q \star \overline{\mathbf{u}_0}(x) - Q \star \overline{\mathbf{u}_0}(y)|. \tag{4.18}$$

Hence, by (3.2), (4.2), (4.11) and (4.12),

$$|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})| \leq C\nu^{-\frac{3}{4}}\varepsilon^{-\frac{5}{3}}\sqrt{W(0)}\left[\varepsilon^{-\frac{1}{3}}\sqrt{W(0)} + 1\right]|\mathbf{x}-\mathbf{y}|^{\frac{1}{2}}t^{\frac{1}{4}} + C'\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}. \tag{4.19}$$

Therefore, the inequality (2.16) page 214 in Leray [27] leads to

$$V_1(t) \leq C\nu^{-\frac{3}{2}}\varepsilon^{-\frac{5}{3}}\sqrt{W(0)}\left[\varepsilon^{-\frac{1}{3}}\sqrt{W(0)} + 1\right]\int_0^t \frac{(t')^{\frac{1}{4}}}{(t-t')^{\frac{3}{4}}} dt' + C'\varepsilon^{-\frac{3}{2}}\sqrt{W(0)}, \tag{4.20}$$

which gives (4.17). □

This result allows us to prove:

Lemma 4.3. *There exists $t_\varepsilon^{(1)}(W(0))$ such that $\forall\, t \in [0, t_\varepsilon^{(1)}(W(0))]$,*

$$J(t) \le 2C\varepsilon^{-1}\sqrt{W(0)}, \quad V_1(t) \le 2C\varepsilon^{-\frac{5}{2}}\sqrt{W(0)}. \tag{4.21}$$

Moreover, the function $x \to t_\varepsilon^{(1)}(x)$ is non increasing.

Proof. Let $t \le t_\varepsilon(W(0))$. On one hand we have

$$\forall\, \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \quad D^\alpha(Q \star \overline{\mathbf{u}_0}) = Q \star D^\alpha \overline{\mathbf{u}_0}. \tag{4.22}$$

On a second hand, we deduce from (3.2), (4.1), (4.2) and the estimate (4.17) in Lemma 4.2 above, that

$$\nabla \mathbf{X} \in L^\infty([0, t_\varepsilon(W(0))] \times \mathbb{R}^3)^{27} \cap L^\infty([0, t_\varepsilon(W(0))], L^2(\mathbb{R}^3)^{27}).$$

This information combined with the estimates (3.16) and (3.18), leads to: $\forall\, t \in [0, t_\varepsilon(W(0))]$,

$$\nabla \mathbf{u}(t, \mathbf{x}) = Q \star \nabla \overline{\mathbf{u}_0} + \int_0^t \int_{\mathbb{R}^3} \nabla \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : \nabla\, \mathbf{X}(t', \mathbf{y})\, d\mathbf{y} dt'. \tag{4.23}$$

The reasoning is very close to that of step ii) of Lemma 3.2's proof so that it is not necessary to repeat it. From (4.23), by combining Young's inequality and (3.18) once again, we have

$$J(t) \le ||\nabla \overline{\mathbf{u}_0}||_{0,2} + C \int_0^t \frac{dt'}{\sqrt{\nu(t - t')}} \left(||\nabla(\overline{\mathbf{u}}(t, \cdot) \otimes \mathbf{u}(t, \cdot))||_{0,2} + ||\nabla \overline{A \nabla \overline{\mathbf{u}}}(t, \cdot)||_{0,2} \right). \tag{4.24}$$

It remains to evaluate each term in the r.h.s of (4.24) one after another, in terms of $J(t)$, without calling on (4.17), that has only served to get (4.23). To begin with, notice that

$$||\nabla \overline{\mathbf{u}_0}||_{0,2} \le C\varepsilon^{-1}\sqrt{W(0)}, \tag{4.25}$$

Furthermore,

$$\begin{aligned} ||\nabla(\overline{\mathbf{u}}(t, \cdot) \otimes \mathbf{u}(t, \cdot))||_{0,2} &\le ||\nabla \overline{\mathbf{u}}(t, \cdot)||_{0,\infty} ||\mathbf{u}(t, \cdot)||_{0,2} + ||\overline{\mathbf{u}}(t, \cdot)||_{0,\infty} J(t) \\ &\le C\varepsilon^{-3/2}\sqrt{W(t)} J(t) \le C\varepsilon^{-3/2}\sqrt{W(0)} J(t), \end{aligned}$$

by using $W(t) \le 4W(0)$ since $t \le t_\varepsilon(W(0))$. Similarly, for the same reason, we also have

$$||\nabla \overline{A \nabla \overline{\mathbf{u}}}(t, \cdot)||_{0,2} \le C\varepsilon^{-2} ||N_A||_\infty \sqrt{W(0)}.$$

Therefore we get

$$J(t) \le C\varepsilon^{-1}\sqrt{W(0)} \left[1 + \int_0^t \frac{\varepsilon^{-1}||N_A||_\infty + \varepsilon^{-1/2} J(t')}{\sqrt{\nu(t - t')}} dt' \right]. \tag{4.26}$$

Using the V-maximum principle once again, we deduce from (4.26) that

$$\forall\, t \in [0, t_\varepsilon^{(1)}(W(0))], \quad J(t) \le 2C\varepsilon^{-1}\sqrt{W(0)}, \tag{4.27}$$

where

$$t_\varepsilon^{(1)}(x) = \inf \left(\frac{\nu \varepsilon^2}{[||N_A||_\infty + 2C\varepsilon^{-1/2}\sqrt{x}]^2}, t_\varepsilon(x) \right). \tag{4.28}$$

We note that $x \to t_\varepsilon^{(1)}(x)$ is non increasing. Once $J(t)$ is under control, we control $V_1(t)$ as in the proof of Lemma 4.1. The estimate (4.21) is now uniform in time and substancially improves (4.17). □

This result yields by induction:

Lemma 4.4. *Let $m \geq 1$. There exists $t_\varepsilon^{(m)}(W(0))$ such that $\forall\, t \in [0, t_\varepsilon^{(m)}(W(0))]$,*

$$||\mathbf{u}(t,\cdot)||_{m,2} \leq C_m \varepsilon^{-m}\sqrt{W(0)}, \quad ||\mathbf{u}(t,\cdot)||_{m,\infty} \leq C_m \varepsilon^{-(m+\frac{3}{2})}\sqrt{W(0)}. \tag{4.29}$$

Moreover, the function $x \to t_\varepsilon^{(m)}(x)$ is non increasing.

Proof. Let $m \geq 1$. The property is already proved in the case $m = 1$ in lemma 4.3. Let $m \geq 2$. For the simplicity, we denote by C the constants instead of C_k $(k = 1, \cdots, m)$. Assume by induction that for any $1 \leq k \leq m-1$ there exists $t_\varepsilon^{(k)}(W(0)) > 0$ such that

$$0 < t_\varepsilon^{(m-1)}(W(0)) \leq \cdots \leq t_\varepsilon^{(k)}(W(0)) \cdots \leq t_\varepsilon(W(0)),$$

and $\forall\, t \in [0, t_\varepsilon^{(k)}(W(0))]$,

$$\begin{aligned} &||\mathbf{u}(t,\cdot)||_{k,2} \leq C\varepsilon^{-k}\sqrt{W(0)}, \\ &||\mathbf{u}(t,\cdot)||_{k,\infty} \leq C\varepsilon^{-(k+\frac{3}{2})}\sqrt{W(0)}, \end{aligned} \tag{4.30}$$

and $\forall\, 1 \leq k \leq m-1$, the function $x \to t_\varepsilon^{(k)}(x)$ is non increasing. We first derive a bound for $||\mathbf{u}(t,\cdot)||_{m,2}$. Before all, we note that:

$$||D^m \overline{\mathbf{u}_0}||_{0,2} \leq C\varepsilon^{-m}\sqrt{W(0)}. \tag{4.31}$$

Let $t \in [0, t_\varepsilon^{(m-1)}(W(0))]$. The same arguments as those developed in the proof of Lemma 4.2 yield by induction to

$$V_m(t) \leq \varphi(t),$$

where $t \to \varphi(t)$ is a continuous function. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = m$. From this, we get

$$D^\alpha \mathbf{X} \in L^\infty([0, t_\varepsilon^{(1)}(W(0))] \times \mathbb{R}^3)^9 \cap L^\infty([0, t_\varepsilon^{(1)}(W(0))], L^2(\mathbb{R}^3)^9),$$

which gives by arguments already set out,

$$D^\alpha \mathbf{u}(t,\mathbf{x}) = Q \star \nabla\overline{\mathbf{u}_0} + \int_0^t \int_{\mathbb{R}^3} \nabla\mathbf{T}(t-t', \mathbf{x}-\mathbf{y}) : D^\alpha \mathbf{X}(t', \mathbf{y})\, d\mathbf{y} dt'.$$

It remains to derive from the induction hypothesis a sharp estimate of $||D^\alpha \mathbf{X}(t', \mathbf{y})||_{0,2}$ in order to use the V-maximum principle to control $||D^\alpha \mathbf{u}(t', \mathbf{y})||_{0,2}$. According to Lemma 4.1, the choice of t and usual results about convolution, we have

$$||D^\alpha \overline{A\nabla\overline{\mathbf{u}}}(t,\cdot)||_{0,2} \leq C||N_A||_\infty \varepsilon^{-(m+1)}\sqrt{W(t)} \leq 2C||N_A||_\infty \varepsilon^{-(m+1)}\sqrt{W(0)}. \tag{4.32}$$

Furthermore, the Leibnitz formula gives,

$$D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u}) = \overline{\mathbf{u}} \otimes D^\alpha \mathbf{u} + \sum_{\substack{\beta=(p,q,r) \\ |\beta|<|\alpha|}} C_{\alpha_1}^p C_{\alpha_2}^q C_{\alpha_3}^r D^\beta \overline{\mathbf{u}} \otimes D^{\alpha-\beta}\mathbf{u}. \tag{4.33}$$

We deduce from (3.2) and Lemma 4.1,

$$||D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u})||_{0,2} \leq C\varepsilon^{-3/2}\sqrt{W(0)}\left[||D^\alpha \mathbf{u}||_{0,2} + \sum_{|\beta|<|\alpha|} \varepsilon^{-|\beta|}||D^{\alpha-\beta}\mathbf{u}||_{0,2}\right].$$

The induction hypothesis yields

$$||D^\alpha(\overline{\mathbf{u}}\otimes\mathbf{u})||_{0,2}\le C\varepsilon^{-3/2}\sqrt{W(0)}\left[||D^\alpha\mathbf{u}||_{0,2}+m\varepsilon^{-2m}\sqrt{W(0)}\right]. \tag{4.34}$$

Notice that we have not optimized things above, and this last estimate could be substancially improved. However, this is not essential. Now, by combining (3.16), (3.18), (4.31), (4.32) and (4.34) we obtain

$$\begin{aligned}&||D^\alpha\mathbf{u}(t,\cdot)||_{0,2}\le C\sqrt{W(0)}\,[\varepsilon^{-m}+\\ &\int_0^t\frac{\varepsilon^{-3/2}||D^\alpha\mathbf{u}(t',\cdot)||_{0,2}+\varepsilon^{-(m+1)}||N_A||_\infty+m\varepsilon^{-(2m+\frac{3}{2})}\sqrt{W(0)}}{\sqrt{\nu(t-t')}}dt'\Bigg].\end{aligned}\tag{4.35}$$

By the V-maximum principle, we deduce from (4.35) that

$$\forall\, t\in[0,t_\varepsilon^{(m)}(W(0))],\quad ||D^\alpha\mathbf{u}(t,\cdot)||_{0,2}\le 2C\varepsilon^{-m}\sqrt{W(0)}, \tag{4.36}$$

where

$$t_\varepsilon^{(m)}(x)=\inf\left(\frac{\nu\varepsilon^2}{\left[||N_A||_\infty+C\varepsilon^{-1/2}(1+m\varepsilon^{-m})\sqrt{x}\right]^2},t_\varepsilon^{(m-1)}(x)\right), \tag{4.37}$$

and $x\to t_\varepsilon^{(m)}(x)$ is clearly non increasing. By a similar process, we also found the bound for $V_m(t)$, therefore $||\mathbf{u}(t,\cdot)||_{m,\infty}$. □

Remark 4.1. *We observe that $t_\varepsilon^{(m)}(x)\to 0$ as $\varepsilon\to 0$, for fixed x and m. Moreover, the only estimate which does not blow up when $\varepsilon\to 0$, is the estimate (4.2) about $W(t)$. This is in coherence with all former known results about the Navier-Stokes equations.*

In the following, we set

$$I_{\varepsilon,m}=[0,t_\varepsilon^{(m)}(W(0))].$$

4.3 Energy balance and transition from local to global time

The transition from local to global time is based on the energy balance, which remains to be justifed. To do so, we must find additional estimates about the pressure to check that it satisfies right integrability conditions. The pressure satisfies the elliptic equation:

$$\Delta p=\mathrm{div}[\mathrm{div}(-\overline{\mathbf{u}}\otimes\mathbf{u}+\overline{A\nabla\overline{\mathbf{u}}})]. \tag{4.38}$$

Therefore, we can write:

$$p(t,\mathbf{x})=\frac{1}{4\pi}\int_{\mathbf{R}^3}\nabla^2\left(\frac{1}{r}\right)[-\overline{\mathbf{u}}\otimes\mathbf{u}+\overline{A\nabla\overline{\mathbf{u}}}](t,\mathbf{y})d\mathbf{y}, \tag{4.39}$$

where $\nabla^2=(\partial_i\partial_j)_{1\le i,j\le 3}$, $r=|\mathbf{x}-\mathbf{y}|$. This expression must be understood as a singular integral operator, with a δ-function for $i=j$. By the Calderón-Zygmund Theorem (see in E. Stein [38] and also in P. Galdi [18], chapters 2 and 3) and Lemma 4.1, we see that $p\in C([0,t_\varepsilon(W(0))],L^2(\mathbb{R}^3))$ and the L^2 bound is uniform in t. From this, the differential quotient method due to L. Nirenberg [32] and the standard elliptic theory (see also in Brezis [7], section IX.6), combined with Lemma 4.4 allows to write by induction that $\forall\, m\ge 0$, $\forall\, t\in I_{\varepsilon,m+2}$,

$$||p(t,\cdot)||_{m,2}\le C(W(0),||N_A||_\infty,\varepsilon,m). \tag{4.40}$$

Having said that, we can be more specific:

Lemma 4.5. *Let $m \geq 4$, $t \in I_{\varepsilon,m+4}$. Then*

$$||p(t,\cdot)||_{m,2} \leq C_m \varepsilon^{-m}\sqrt{W(0)}\left[\varepsilon^{-1}||N_A||_\infty + \varepsilon^{-m}\sqrt{W(0)}\right], \tag{4.41}$$

where we recall that $N_A(t) = ||A(t,\cdot)||_{0,\infty}$.

Proof. Let $t \in I_{\varepsilon,m+4}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| \leq m$. As $H^4(\mathbb{R}^3) \hookrightarrow C^2(\mathbb{R}^3)$, then $D^\alpha \mathbf{u}(t,\cdot)$, $D^\alpha p(t,\cdot) \in C^2(\mathbb{R}^3)$, which allows to derive the regularized Navier-Stokes equations up to order α. In particular, we have at time t:

$$\partial_t D^\alpha \mathbf{u} - \nu \Delta D^\alpha \mathbf{u} + \nabla D^\alpha p = -D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u}) + D^\alpha(\overline{A\nabla\overline{\mathbf{u}}}). \tag{4.42}$$

Subsequently, from $\text{div} D^\alpha \mathbf{u} = 0$ we get,

$$D^\alpha p(t,\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \nabla^2\left(\frac{1}{r}\right)[D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u})(t,\mathbf{y}) - D^\alpha(\overline{A\nabla\overline{\mathbf{u}}})(t,\mathbf{y})]d\mathbf{y}. \tag{4.43}$$

As $m \geq 4$, $H^m(\mathbb{R}^3)$ is an algebra, hence by (4.29)

$$||(\overline{\mathbf{u}} \otimes \mathbf{u})(t,\cdot)||_{m,2} \leq C||\overline{\mathbf{u}}(t,\cdot)||_{m,2}||\mathbf{u}(t,\cdot)||_{m,2} \leq C_m \varepsilon^{-2m} W(0). \tag{4.44}$$

Therefore, combining this inequality with (4.32) and Calderòn-Zigmung Theorem, we obtain

$$\begin{aligned} ||D^\alpha p(t,\cdot)||_{0,2} &\leq C(||D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u})(t,\cdot)||_{0,2} + ||D^\alpha \overline{A\nabla\overline{\mathbf{u}}}(t,\cdot)||_{0,2}) \\ &\leq C_m \varepsilon^{-m}\sqrt{W(0)}\left[\varepsilon^{-m}\sqrt{W(0)} + \varepsilon^{-1}||N_A||_\infty\right] = P_{m,\varepsilon}. \end{aligned} \tag{4.45}$$

Therefore, $p(t,\cdot) \in H^m(\mathbb{R}^3)$ for all $t \in I_{\varepsilon,m+4}$ and

$$||p(t,\cdot)||_{m,2} \leq m P_{m,\varepsilon}, \tag{4.46}$$

giving (4.41). □

Lemma 4.6. *Let $m \geq 4$. Then $\mathbf{u} \in C(I_{\varepsilon,m+4}, H^m(\mathbb{R}^3)^3)$ and for all α such that $|\alpha| \leq m$, the following energy balance holds:*

$$\frac{1}{2}\int_{\mathbf{R}^3} |D^\alpha \mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} = \frac{1}{2}\int_{\mathbf{R}^3} |D^\alpha \overline{\mathbf{u}_0}(\mathbf{x})|^2 d\mathbf{x} + \int_0^t \left[I_1(t') + I_2(t') + I_3(t')\right] dt', \tag{4.47}$$

where

$$\begin{cases} I_1(t') = -\displaystyle\int_{\mathbf{R}^3} D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u})(t',\mathbf{x}) : \nabla D^\alpha \mathbf{u}(t',\mathbf{x}) d\mathbf{x}, \\ I_2(t') = -\nu \displaystyle\int_{\mathbf{R}^3} |\nabla D^\alpha \mathbf{u}(t',\mathbf{x})|^2 d\mathbf{x}, \\ I_3(t') = -\displaystyle\int_{\mathbf{R}^3} D^\alpha \overline{A\nabla\overline{\mathbf{u}}}(t',\mathbf{x}) \cdot \nabla D^\alpha \mathbf{u}(t',\mathbf{x}) d\mathbf{x}. \end{cases}$$

In particular, we have $\forall\, t \in I_{\varepsilon,m+4}$,

$$\frac{1}{2}W(t) + \nu\int_0^t J^2(t')dt' + \int_0^t K_{A,\varepsilon}^2(t')dt' = \frac{1}{2}W_\varepsilon(0), \tag{4.48}$$

for all $t \in [0,T[$, where $W(t)$ and $J(t)$ were initially defined by (2.1) and (2.2), and $K_{A,\varepsilon}$ by (3.8).

Proof. We deduce from the equation (4.42),

$$D^\alpha(\partial_t \mathbf{u}) = -\mathrm{div}(D^\alpha(\overline{\mathbf{u}} \otimes \mathbf{u})) + \nu \Delta D^\alpha \mathbf{u} + \mathrm{div}\, D^\alpha \overline{A \nabla \overline{\mathbf{u}}} - \nabla D^\alpha p. \tag{4.49}$$

Therefore, by (4.29) and (4.41), we get

$$\begin{aligned} &||D^\alpha(\partial_t \mathbf{u})(t,\cdot)||_{0,2} \leq \\ &\qquad C_{m+1}\varepsilon^{-(m+1)}\sqrt{W(0)}\left[\sqrt{W(0)}(1+\varepsilon^{-m}) + \varepsilon^{-1}(\nu + ||N_A||_\infty)\right]. \end{aligned} \tag{4.50}$$

By consequence, $\mathbf{u}, \partial_t \mathbf{u} \in L^\infty(I_{\varepsilon,m+4}, H^m(\mathbb{R}^3)^3) \subset L^2(I_{\varepsilon,m+4}, H^m(\mathbb{R}^3)^3)$. Therefore, by a well known result of functional analysis (see in Temam [42] for example), $\mathbf{u} \in C(I_{\varepsilon,m+4}, H^m(\mathbb{R}^3)^3)$ and

$$(\partial_t \mathbf{u}(t,\cdot), \mathbf{u}(t,\cdot))_m = \frac{d}{2dt}||\mathbf{u}(t,\cdot)||^2_{m,2}.$$

Following the usual process, we form the dot product of the equation (4.49) with $D^\alpha \mathbf{u}$. We integrate over $\mathbb{R}^3$ by using the Stokes formula, which is possible by the integrabilty properties of $D^\alpha \mathbf{u}$ and $D^\alpha p$. In particular, since $\mathrm{div} D^\alpha \mathbf{u} = 0$, we get

$$(\nabla D^\alpha p, D^\alpha \mathbf{u}) = 0.$$

We get the energy balance (4.47) after integrating in time over $[0,t]$. In the special case $\alpha = 0$, we obtain (4.48) by noting that in addition: $(\overline{\mathbf{u}} \cdot \nabla \mathbf{u}, \mathbf{u}) = 0$. $\square$

Theorem 4.1. *Let $T > 0$. Then*[3] $\forall\, m \geq 4$, $(\mathbf{u}, p) \in C([0,T], H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3))$, *the energy balance (4.48) holds for all $t \in [0,T]$ as well as the estimates (4.29) and (4.41).*

Proof. The energy balance (4.48) combined with the inequality (3.7), shows that for all $t \in I_{\varepsilon,m+4}$, we have $W(t) \leq W(0)$, improving substancially the estimate (4.1). In particular we can write

$$W(\delta t_\varepsilon^{(m)}(W(0)) \leq W(0), \quad \text{where} \quad \delta t_\varepsilon^{(m)}(x) = t_\varepsilon^{(m+4)}(x). \tag{4.51}$$

Let $(t_{n,\varepsilon}^{(m)})_{n\geq 1}$ be the sequence given by

$$t_{1,\varepsilon}^{(m)} = \delta t_\varepsilon^{(m)}(W(0)), \quad t_{n,\varepsilon}^{(m)} = t_{n-1,\varepsilon}^{(m)} + \delta t_\varepsilon^{(m)}(W(t_{n-1,\varepsilon})). \tag{4.52}$$

Assume that for a given n, $t_{n,\varepsilon}^{(m)}$ is constructed such that $u \in C([0, t_{n,\varepsilon}^{(m)}], H^m(\mathbb{R}^3))$, which holds when $n = 1$ by Lemma 4.4 since $t_\varepsilon^{(m+4)} \leq t_\varepsilon^{(m)}$. In particular the energy balance holds over $[0, t_{n,\varepsilon}^{(m)}]$, and $W(t_{n,\varepsilon}^{(m)}) \leq W(0)$, which yields $\delta t_\varepsilon^{(m)}(W(t_{n,\varepsilon}^{(m)})) \geq \delta t_\varepsilon^{(m)}(W(0))$ by the decrease of the function $x \to t_\varepsilon^{(m+4)}(x)$. Therefore, we can reproduce the arguments of the section 4.1 and the lemma 4.5 and 4.6 from the time $t_{n,\varepsilon}^{(m)}$, by the continuity in time, thus validating the iteration $n+1$ of (4.52) by the inductive hypothesis. In particular we have

$$t_{n+1,\varepsilon}^{(m)} = \sum_{k=1}^{n} \delta t_\varepsilon^{(m)}(W(t_{k,\varepsilon}^{(m)})) \geq n\delta t_\varepsilon^{(m)}(W(0)),$$

which is larger than T for n large enough, concluding the proof. $\square$

[3]we take $m \geq 4$ to be in coherence with the arguments set out in Lemma 4.6. However, what can do more can do less, and the result holds for $m = 0, 1, 2, 3$.

Remark 4.2. *The estimates obtained in this section prove that for any $m \geq 0$,*

$$||\mathbf{u}(t,\cdot) - \overline{\mathbf{u}_0}||_{m,2} \leq C_m \varepsilon^{-(2m+\frac{3}{2})} \sqrt{\frac{t}{\nu}}. \tag{4.53}$$

Remark 4.3. *The continuity in time ensures that any regular solution to the regularized NSE (3.5) satisfies the following semi-group like property: $\forall\, t_0 \in]0,T[$, $\forall\, t \in [t_0, T[$,*

$$\begin{cases} \mathbf{u}(t,\mathbf{x}) = (Q \star \mathbf{u}(t_0,\cdot))(t,\mathbf{x}) \\ \\ + \displaystyle\int_{t_0}^t \int_{\mathbf{R}^3} \nabla \mathbf{T}(t-t', \mathbf{x}-\mathbf{y}) : [\overline{\mathbf{u}}(t',\mathbf{y}) \otimes \mathbf{u}(t',\mathbf{y}) - \overline{A\nabla\overline{\mathbf{u}}}(t',\mathbf{y})]\, d\mathbf{y}dt'. \end{cases} \tag{4.54}$$

This what is implicitely used in the proof of Theorem 4.1.

Once the regularity result of Theorem 4.1 is established, following the standard routine yields the uniqueness result:

Theorem 4.2. *The regularized NSE (3.5) have at the most one regular soution $(\mathbf{u}, p)$.*

5 Existence of solution for the regularized NSE

The aim of this section is the proof of the existence result stated in Theorem 3.1. The solution of the regularized NSE (3.5) is constructed by a standard Picard iteration process based on the Oseen integral representation.

5.1 Iterations

Let us put

$$\mathbf{u}_1(t,\mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t,\mathbf{x}), \tag{5.1}$$

and for all $n > 0$,

$$\begin{cases} \mathbf{u}_{n+1}(t,\mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t,\mathbf{x}) \\ \\ + \displaystyle\int_0^t \int_{\mathbf{R}^3} \nabla \mathbf{T}(t-t', \mathbf{x}-\mathbf{y}) : [(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n)(t',\mathbf{y}) - \overline{A\nabla\overline{\mathbf{u}}_n}(t',\mathbf{y})]\, d\mathbf{y}dt'. \end{cases} \tag{5.2}$$

The first result of this section aims to check that the sequence $(\mathbf{u}_n)_{n\in\mathbf{N}}$ makes sense and to get estimates about $\mathbf{u}_n$, similar to the estimates (4.29).

Lemma 5.1. *for all $m \geq 0$, $n \geq 1$, $\mathbf{u}_n \in C(I_{\varepsilon,m}, H^m(\mathbb{R}^3))^3$, where $I_{\varepsilon,m} = [0, t_\varepsilon^{(m)}(W(0))]$, $t_\varepsilon^{(m)}(x)$ is given by (4.37). Moreover, we have, $\forall\, m \geq 0$, $\forall\, t \in [0, t_\varepsilon^{(m)}(W(0))]$, $\forall\, n \in \mathbb{N}$:*

$$||\mathbf{u}_n(t,\cdot)||_{m,2} \leq C_m \varepsilon^{-m} \sqrt{W(0)}, \quad ||\mathbf{u}_n(t,\cdot)||_{m,\infty} \leq C_m \varepsilon^{-(m+\frac{3}{2})} \sqrt{W(0)}. \tag{5.3}$$

Proof. By recycling the proof of Lemma 4.1, we get the inequality

$$\sqrt{W_{n+1}(t)} \leq \sqrt{W(0)} + C\varepsilon^{-1} \int_0^t \frac{\varepsilon^{-1/2} W_n(t') + \sqrt{W_n(t')}}{\sqrt{\nu(t-t')}} dt'. \tag{5.4}$$

A straightforward inductive reasonning yields

$$\forall\, t \in [0, t_\varepsilon(W(0))], \quad \forall n \in \mathbb{N}, \quad W_n(t) \leq 4W(t), \tag{5.5}$$

where $t_\varepsilon(x)$ is specified by the formula (4.9), in which we take $T = \infty$. Following the proofs of Lemma 4.3 and 4.4 we obtain (5.3). We skip the details. The continuity in time is obtained in the same way as in the item 2) of the proof of Lemma 3.2, based on $\nabla \mathbf{T} \in L^1_{t,\mathbf{x}}$. $\square$

Lemma 5.2. *Let $m \geq 4$, $n \geq 0$. There exists $p_{n+1} \in C(I_{\varepsilon,m+4}, H^m(\mathbb{R}^3))$ such that $(\mathbf{u}_{n+1}, p_{n+1})$ satisfies over $I_{\varepsilon,m+4} \times \mathbb{R}^3$ the evolutionary Stokes equation:*

$$\begin{cases} \partial_t \mathbf{u}_{n+1} - \nu \Delta \mathbf{u}_{n+1} + \nabla p_{n+1} = -\mathrm{div}(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n - \overline{A \nabla \mathbf{u}_n}), \\ \mathrm{div}\, \mathbf{u}_{n+1} = 0, \\ \mathbf{u}_{n+1}|_{t=0} = \overline{\mathbf{u}_0}. \end{cases} \tag{5.6}$$

Proof. Let $\mathbf{X}_n = \overline{\mathbf{u}}_n \otimes \mathbf{u}_n - \overline{A \nabla \mathbf{u}_n}$, and consider the evolutionary Stokes problem

$$\begin{cases} \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p_{n+1} = -\mathrm{div} \mathbf{X}_n, \\ \mathrm{div}\, \mathbf{v} = 0, \\ \mathbf{v}|_{t=0} = \overline{\mathbf{u}_0}. \end{cases} \tag{5.7}$$

According to Lemma 5.1 and because $m \geq 4$, we have at least $\mathbf{X}_n \in L^2(I_{\varepsilon,m+4}, H^3(\mathbb{R}^3))$. Then by Theorem 1.1 and Proposition 1.2 in Chapter 3 in Temam [42], we know the existence of a unique weak solution to the Stokes problem (5.7) such that

$$\partial_t \mathbf{u} \in L^2(I_{\varepsilon,m+4}, H^3(\mathbb{R}^3)), \quad \mathbf{u} \in L^2(I_{\varepsilon,m+4}, H^5(\mathbb{R}^3)), \quad p_{n+1} \in L^2(I_{\varepsilon,m+4}, H^4(\mathbb{R}^3)),$$

which is constructed by the Galerkin method. Therefore, the conditions for the application of Lemma 8 in Leray [27] are fulfiled, and

$$\begin{cases} \mathbf{v}(t, \mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t, \mathbf{x}) \\ \\ + \displaystyle\int_0^t \int_{\mathbb{R}^3} \nabla \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : [(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n)(t', \mathbf{y}) - \overline{A \nabla \overline{\mathbf{u}}_n}(t', \mathbf{y})] \, d\mathbf{y} dt', \end{cases} \tag{5.8}$$

hence $\mathbf{v} = \mathbf{u}_{n+1}$ because the solution of (5.7) is unique. From there, the regularity of p_{n+1} is obtained by the same argument as in the proof of (4.41). $\square$

5.2 Contraction property

In what follows, we set

$$W_n(t) = \int_{\mathbb{R}^3} |\mathbf{u}_n(t, \mathbf{x})|^2 d\mathbf{x} = ||\mathbf{u}_n(t, \cdot)||^2_{0,2}, \tag{5.9}$$

$$V_n(t) = ||\mathbf{u}_n(t, \cdot)||_{0,\infty}, \tag{5.10}$$

$$J_n(t) = ||\nabla \mathbf{u}_n(t, \cdot)||_{0,2}, \tag{5.11}$$

$$V_{m,n}(t) = \sup_{\mathbf{x} \in \mathbb{R}^3} |D^m \mathbf{u}(t, \mathbf{x})| = ||D^m \mathbf{u}(t, \cdot)||_{0,\infty}. \tag{5.12}$$

We prove in this section that there exists $\tau_\varepsilon^{(m)}(W(0)) > 0$ such that the sequence $(\mathbf{u}_n)_{n \in \mathbb{N}}$ satisfies a contraction property over $[0, \tau_\varepsilon^{(m)}(W(0))]$. Given any time $\tau > 0$, we equip the space $C([0, \tau], H^m(\mathbb{R}^3))$ with the natural uniform norm

$$||w||_{\tau;m,2} = \sup_{t \in [0,\tau]} ||w(t, \cdot)||_{m,2}, \tag{5.13}$$

making $C([0, \tau], H^m(\mathbb{R}^3))$ a Banach space. We show in this subsection the following

Lemma 5.3. *For all $m \geq 0$, there exists a time $\tau_\varepsilon^{(m)}(W(0))$ such that*

$$\forall\, n \in \mathbb{N}, \quad ||\mathbf{u}_{n+1} - \mathbf{u}_n||_{\tau_\varepsilon^{(m)}(W(0));m,2} \leq \frac{1}{2}||\mathbf{u}_n - \mathbf{u}_{n-1}||_{\tau_\varepsilon^{(m)}(W(0));m,2}. \tag{5.14}$$

Moreover, the function $x \to \tau_\varepsilon^{(m)}(x)$ is non increasing.

Proof. Let $m \geq 0$. From now, we are working on the time interval $I_{\varepsilon,m+4} = [0, t_\varepsilon^{(m+4)}(W(0))]$. Let $n \geq 1$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| \leq m$, and $w_{n,\alpha}(t)$ defined by

$$w_{n,\alpha}(t) = ||D^\alpha \mathbf{u}_n(t,\cdot) - D^\alpha \mathbf{u}_{n-1}(t,\cdot)||_{0,2}. \tag{5.15}$$

We first evaluate $w_{n+1,0}(t)$ in terms of $w_{n,0}(t)$. To do so, let

$$\Delta_{n,0}(t') = ||(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n)(t',\cdot) - (\overline{\mathbf{u}}_{n-1} \otimes \mathbf{u}_{n-1})(t',\cdot)||_{0,2},$$

and observe that at any time t', we have

$$\Delta_{n,0}(t') \leq ||\overline{\mathbf{u}}_n(t',\cdot)||_{0,\infty} w_{n,0}(t') + ||\mathbf{u}_{n-1}(t',\cdot)||_{0,\infty}||(\overline{\mathbf{u}}_n - \overline{\mathbf{u}}_{n-1})(t',\cdot)||_{0,2}, \tag{5.16}$$

which leads by (5.3) to

$$\Delta_{n,0}(t') \leq C\sqrt{W(0)}w_{n,0}(t'). \tag{5.17}$$

Similarly, we also have

$$||\overline{A\nabla\mathbf{u}_n}(t',\cdot) - \overline{A\nabla\mathbf{u}_{n-1}}(t',\cdot)||_{0,2} \leq C||N_A||_\infty \varepsilon^{-1} w_{n,0}(t'). \tag{5.18}$$

Inequalities (5.17) and (5.18) combined with the relation (5.2) and arguments used many times before lead to

$$w_{n+1,0}(t) \leq C(\sqrt{W(0)} + \varepsilon^{-1}||N_A||_\infty) \int_0^t \frac{w_{n,0}(t')}{\sqrt{t-t'}}dt'. \tag{5.19}$$

The same procedure leads to: $\forall\, \alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| \leq m$,

$$w_{n+1,\alpha}(t) \leq C_{n,\alpha,\varepsilon}(\sqrt{W(0)} + ||N_A||_\infty) \int_0^t \frac{||\mathbf{u}_n(t',\cdot) - \mathbf{u}_{n-1}(t',\cdot)||_{m,2}}{\sqrt{\nu(t-t')}}dt'. \tag{5.20}$$

To see this, we must first estimate $\Delta_{n,\alpha}(t')$, where

$$\Delta_{n,\alpha}(t') = ||D^\alpha(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n)(t',\cdot) - D^\alpha(\overline{\mathbf{u}}_{n-1} \otimes \mathbf{u}_{n-1})(t',\cdot)||_{0,2}.$$

By the Leibnitz formula, we have

$$D^\alpha(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n) - D^\alpha(\overline{\mathbf{u}}_{n-1} \otimes \mathbf{u}_{n-1}) =$$
$$\sum_{\substack{\beta=(p,q,r)\\ |\beta|\leq|\alpha|}} C_{\alpha_1}^p C_{\alpha_2}^q C_{\alpha_3}^r (D^\beta \overline{\mathbf{u}}_{n+1} \otimes D^{\alpha-\beta}\mathbf{u}_{n+1} - D^\beta \overline{\mathbf{u}}_n \otimes D^{\alpha-\beta}\mathbf{u}_n).$$

We deduce from inequality (5.3) that

$$\begin{aligned} &||(D^\beta \overline{\mathbf{u}}_{n+1} \otimes D^{\alpha-\beta}\mathbf{u}_{n+1})(t',\cdot) - (D^\beta \overline{\mathbf{u}}_n \otimes D^{\alpha-\beta}\mathbf{u}_n)(t',\cdot)||_{0,2} \leq \\ &\qquad \sqrt{W(0)}\left[C_\beta \varepsilon^{-|\beta|} w_{n,\alpha-\beta}(t') + C_{\alpha-\beta}\varepsilon^{-(|\alpha|-|\beta|)} w_{n,\beta}(t')\right]. \end{aligned} \tag{5.21}$$

Furthermore,

$$||D^\alpha \overline{A\nabla\overline{\mathbf{u}}_n}(t',\cdot) - D^\alpha \overline{A\nabla\overline{\mathbf{u}}_{n-1}}(t',\cdot)||_{0,2} \le C||N_A||_\infty \varepsilon^{-(|\alpha|+1)} w_{n,0}(t'). \tag{5.22}$$

By noting that $\forall\,\beta$ s.t. $|\beta| \le m$,

$$w_{n,\beta}(t') \le ||\mathbf{u}_n(t',\cdot) - \mathbf{u}_{n-1}(t',\cdot)||_{m,2},$$

we get (5.20) by combining (5.21), (5.22) and (5.2).
Summing (5.20) over α for $0 \le |\alpha| \le m$, yields for all $t \in I_{\varepsilon,m}$,

$$||\mathbf{u}_{n+1}(t,\cdot) - \mathbf{u}_n(t,\cdot)||_{m,2} \le \\ C_{n,\alpha,\varepsilon}(\sqrt{W(0)} + ||N_A||_\infty) \int_0^t \frac{||\mathbf{u}_n(t',\cdot) - \mathbf{u}_{n-1}(t',\cdot)||_{m,2}}{\sqrt{\nu(t-t')}} dt',$$

hence $\forall\,\tau \in I_{\varepsilon,m}$, $\forall\,t \in [0,\tau]$,

$$||\mathbf{u}_{n+1}(t,\cdot) - \mathbf{u}_n(t,\cdot)||_{m,2} \le \\ C_{n,\alpha,\varepsilon}(\sqrt{W(0)} + ||N_A||_\infty) \sup_{t' \in [0,\tau]} ||\mathbf{u}_n(t',\cdot) - \mathbf{u}_{n-1}(t',\cdot)||_{m,2} \sqrt{\frac{\tau}{\nu}}.$$

Consequently, (5.14) holds by taking

$$\tau_\varepsilon^{(m)}(x) = \inf\left(\frac{\nu}{2C_{n,\alpha,\varepsilon}(\sqrt{x} + ||N_A||_\infty)^2}, t_{m+4,\varepsilon}(x) \right). \tag{5.23}$$

which is indeed a non increasing function of x. □

5.3 Concluding proof

We are now capable of proving Theorem 3.1. Let $m \ge 4$, so that $H^m(\mathbb{R}^3) \hookrightarrow C^2(\mathbb{R}^3)$. For the simplicity we write τ instead of $\tau_\varepsilon^{(m)}(W(0))$. Lemma 5.3 shows that the sequence $(\mathbf{u}_n)_{n\in\mathbf{N}}$ converges to some $\mathbf{u}$ in $C([0,\tau], H^m(\mathbb{R}^3)^3)$. We aim to prove that $\mathbf{u}$ satisfies the Oseen integral relation (3.22). Let $t \in [0,\tau]$, and consider

$$\begin{aligned} v_n(t,\mathbf{x};t') &= \int_{\mathbf{R}^3} \nabla\mathbf{T}(t-t',\mathbf{x}-\mathbf{y}) : [(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n)(t',\mathbf{y}) - \overline{A\nabla\overline{\mathbf{u}}_n}(t',\mathbf{y})]\, d\mathbf{y}, \\ v(t,\mathbf{x};t') &= \int_{\mathbf{R}^3} \nabla\mathbf{T}(t-t',\mathbf{x}-\mathbf{y}) : [(\overline{\mathbf{u}} \otimes \mathbf{u})(t',\mathbf{y}) - \overline{A\nabla\overline{\mathbf{u}}}(t',\mathbf{y})]\, d\mathbf{y}. \end{aligned} \tag{5.24}$$

The inequality (3.16) leads to, for $t' < t$,

$$|v_n(t,\mathbf{x};t') - v(t,\mathbf{x};t')| \le \\ \frac{C}{[\nu(t-t')]^2} \int_{\mathbf{R}^3} \left[|(\overline{\mathbf{u}}_n \otimes \mathbf{u}_n - \overline{\mathbf{u}} \otimes \mathbf{u})(t',\mathbf{y})| + |(\overline{A\nabla\overline{\mathbf{u}}_n} - \overline{A\nabla\overline{\mathbf{u}}})(t',\mathbf{y})| \right] d\mathbf{y},$$

which ensures, by the uniform convergence of $(\mathbf{u}_n(t',\cdot))_{n\in\mathbf{N}}$ to $\mathbf{u}(t',\cdot)$ in $H^m(\mathbb{R}^3)^3$, that for any fixed $(t,\mathbf{x}) \in [0,\tau] \times \mathbb{R}^3$, any $t' \in [0,t[$,

$$v_n(t,\mathbf{x};t') \to v(t,\mathbf{x};t') \quad \text{as} \quad n \to \infty.$$

Moreover, the inequality

$$|v_n(t,\mathbf{x};t')| \le \left(V_{n,0}(t')^2 + ||N_A||_\infty V_{n,1}(t')\right) ||\nabla\mathbf{T}(t',\cdot)||_{0,1}, \tag{5.25}$$

gives by (3.18) and (5.3),

$$|v_n(t,\mathbf{x};t')| \leq \frac{C\varepsilon^{-3/2}[W(0)+\varepsilon^{-1}||N_A||_\infty]}{\sqrt{\nu(t-t')}} \in L^1([0,t]).$$

Therefore, Lebesgue's Theorem applies and we have

$$\int_0^t v_n(t,\mathbf{x};t')dt' \to \int_0^t v(t,\mathbf{x};t')dt' \text{ as } n\to\infty,$$

in other words, $\mathbf{u}$ satisfies the integral relation (3.22) and the regularity results of Section 4 apply for $\mathbf{u}$. By the same proof as that of Lemma 5.2, we see that there exists a scalar field p such that $(\mathbf{u},p)$ is a regular solution to the regularized NSE (1.3), over the time interval $[0,\tau_\varepsilon^{(m)}(W(0))]$. The transition from local to global time is like in Theorem 4.1's proof, by the decrase of the function $x\to\tau_\varepsilon^{(m)}(x)$, the time continuity of the trajectories in H^m and the energy balance. The proof of Theorem 3.1 is now completed. □

5.4 Behavior at infinity

In order to take the limit in the regularized NSE when $\varepsilon\to 0$, we need to know how the kinetic energy of the velocity field $\mathbf{u}$ behaves for large values of $|\mathbf{x}|$. From now, we assume that A is of compact support uniformly in t, which means that there exists R_0 verifying:

$$\forall\,\mathbf{x}\in\mathbb{R}^3 \text{ s.t. } |\mathbf{x}|\geq R_0,\quad \forall\,t\geq 0,\quad A(t,\mathbf{x})=0. \tag{5.26}$$

We prove in this subsection:

Lemma 5.4. *There exists a non increasing continous function of t, $\varphi=\varphi(t)$, such that for any $R_1>0$, $R_2>0$, $R_1<R_2$,*

$$\frac{1}{2}\int_{|\mathbf{x}|\geq R_2}|\mathbf{u}(t,\mathbf{x})|^2d\mathbf{x} \leq \frac{1}{2}\int_{|\mathbf{x}|\geq R_1}|\mathbf{u}_0(\mathbf{x})|^2d\mathbf{x}+\frac{1}{R_2-R_1}\varphi(t). \tag{5.27}$$

Proof. Let R_1, R_2, $0<R_1<R_2$, and $f=f(\mathbf{x})$ be the function defined by

$$\begin{array}{ll} f(\mathbf{x})=0 & \text{if } |\mathbf{x}|\leq R_1,\\ f(\mathbf{x})=\dfrac{|\mathbf{x}|-R_1}{R_2-R_1} & \text{if } R_1\leq|\mathbf{x}|\leq R_2,\\ f(\mathbf{x})=1 & \text{if } |\mathbf{x}|\geq R_2. \end{array} \tag{5.28}$$

Taking $f(\mathbf{x})\mathbf{u}(t,\mathbf{x})$ as test in (3.5.i) and integrating by parts by using $\operatorname{div}\mathbf{u}=0$, yields at each time t,

(5.29)

$$\left\{\begin{array}{l} \dfrac{1}{2}\displaystyle\int_{\mathbf{R}^3} f(\mathbf{x})|\mathbf{u}(t,\mathbf{x})|^2d\mathbf{x}+\nu\int_0^t\int_{\mathbf{R}^3} f(\mathbf{x})|\nabla\mathbf{u}(t',\mathbf{x})|^2d\mathbf{x}dt' = \\ \dfrac{1}{2}\displaystyle\int_{\mathbf{R}^3} f(\mathbf{x})|\overline{\mathbf{u}_0}(\mathbf{x})|^2d\mathbf{x}-\nu\int_0^t\int_{\mathbf{R}^3}\nabla\mathbf{u}(t',\mathbf{x})\cdot\nabla f(\mathbf{x})\cdot\mathbf{u}(t',\mathbf{x})\,d\mathbf{x}dt'+ \\ \displaystyle\int_0^t\int_{\mathbf{R}^3}\mathbf{u}(t',\mathbf{x})\cdot\nabla f(\mathbf{x})\,p(t',\mathbf{x})d\mathbf{x}dt'+\frac{1}{2}\int_0^t\int_{\mathbf{R}^3}\overline{\mathbf{u}}(t',\mathbf{x})\cdot\nabla f(\mathbf{x})\,|\mathbf{u}(t',\mathbf{x})|^2d\mathbf{x}dt'- \\ -\displaystyle\int_0^t\int_{\mathbf{R}^3}A(t',\mathbf{x})\nabla\overline{\mathbf{u}}(t',\mathbf{x}):\nabla(\overline{f(\mathbf{x})\mathbf{u}(t',\mathbf{x})})d\mathbf{x}dt' \end{array}\right.$$

Taking $R_1 \geq R_0$ where R_0 is specified in (5.30), leads to

$$\int_0^t \int_{\mathbf{R}^3} A(t', \mathbf{x}) \nabla \overline{\mathbf{u}}(t', \mathbf{x}) : \nabla (\overline{f(\mathbf{x}) \mathbf{u}(t', \mathbf{x})}) d\mathbf{x} dt' = 0. \tag{5.30}$$

From there, the calculations carried out in Leray [27], section 27, pages 232-234, can be directly reused, and the conclusion follows. □

Remark 5.1. *The assumption "A is with compact support" is consistent with the idea that no turbulence occurs for large values of $|\mathbf{x}|$, which is in agreement with the results of Caffarelli-Kohn-Nirenberg [8] about the singularitie's location (if any) of NSE's solutions without eddy viscosity, so far we believe that turbulence and singularities are connected. We conjecture that this is not needed, which is leaved as an open question.*

6 Passing to the limit in the equations

6.1 Aim

from now, $\varepsilon > 0$ being given, $(\mathbf{u}_\varepsilon, p_\varepsilon)$ denotes the solution to the regularized NSE (3.5). The aim of this section is to show that we can extract from $(\mathbf{u}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ a subsequence that converges to a turbulent solution of the NSE (1.1) (see Definition 2.3), which will prove Theorem 2.3. We follow roughly speaking the frame set out by J. Leray to pass to the limit. We have filled in the blanks, refreshed and customized this frame by using modern tools of analysis, taking into account the eddy viscosity term that is not in Leray's work.

Recall that the assumptions about A are given by items i)), ii)) and ii)) in the statement of Theorem 2.3. We also recall that the space of test vector fields E_σ we are considering is given by

$$\begin{aligned} \mathrm{E}_\sigma = \Big\{ &\mathbf{w} \in L^1_{loc}(\mathbb{R}_+, H^3(\mathbb{R}^3)^3) \quad \text{s.t.} \quad \mathbf{w} \in C(\mathbb{R}^+, L^2(\mathbb{R}^3)^3), \\ &\nabla \mathbf{w} \in L^\infty(\mathbb{R}, C_b(\mathbb{R}^3)^3), \quad \frac{\partial \mathbf{w}}{\partial t} \in L^1_{loc}(\mathbb{R}_+, L^2(\mathbb{R}^3)^3), \quad \operatorname{div} \mathbf{w} = 0 \Big\}. \end{aligned} \tag{6.1}$$

Let $\mathbf{w} \in \mathrm{E}_\sigma$. We form the scalar product of $\mathbf{w}$ with the momentum equation (3.5.i) and integrate by parts, which is legitimate by the regularities of $\mathbf{u}_\varepsilon$, p_ε and $\mathbf{w}$. We get:

$$\left\{ \begin{aligned} &\int_{\mathbf{R}^3} \overline{\mathbf{u}_0}(\mathbf{x}) \cdot \mathbf{w}(0, \mathbf{x}) d\mathbf{x} = \int_{\mathbf{R}^3} \mathbf{u}_\varepsilon(t, \mathbf{x}) \cdot \mathbf{w}(t, \mathbf{x}) d\mathbf{x} \\ &- \int_0^t \int_{\mathbf{R}^3} \mathbf{u}_\varepsilon(t', \mathbf{x}) \cdot \left[\nu \Delta \mathbf{w}(t', \mathbf{x}) + \operatorname{div}(\overline{A \nabla \overline{\mathbf{w}}})(t', \mathbf{x})) + \frac{\partial \mathbf{w}}{\partial t'}(t', \mathbf{x}) \right] d\mathbf{x} dt' \\ &+ \int_0^t \int_{\mathbf{R}^3} [\overline{\mathbf{u}_\varepsilon}(t', \mathbf{x}) \otimes \mathbf{u}_\varepsilon(t', \mathbf{x})] : \nabla \mathbf{w}(t', \mathbf{x}) \, d\mathbf{x} dt', \end{aligned} \right. \tag{6.2}$$

where also have used:

$$\begin{aligned} \int_{\mathbf{R}^3} \operatorname{div}(\overline{A \nabla \overline{\mathbf{u}}}) \cdot \mathbf{w} = - \int_{\mathbf{R}^3} \overline{A \nabla \overline{\mathbf{u}}} : \nabla \mathbf{w} = - \int_{\mathbf{R}^3} A \nabla \overline{\mathbf{u}} : \overline{\mathbf{w}} = \\ - \int_{\mathbf{R}^3} \nabla \mathbf{u} : \overline{A \overline{\mathbf{w}}} = \int_{\mathbf{R}^3} \mathbf{u} \cdot \operatorname{div}(\overline{A \overline{\mathbf{w}}}). \end{aligned}$$

The goal is to study how to pass to the limit in (6.2) when $\varepsilon \to 0$. We note that the only avaible estimates which do not depend on ε are those given by the energy balance

(3.6), which shows that the sequence $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$ as well as in $L^2(\mathbb{R}_+, H^1(\mathbb{R}^3)^3$. A little bit more can be said:

Lemma 6.1. *i) The function $t \to W_\varepsilon(t)$ is uniformly bounded, namely $\forall\, \varepsilon > 0$, $\forall\, t \geq 0$,*

$$W_\varepsilon(t) \leq W_\varepsilon(0) \leq W(0), \tag{6.3}$$

ii) For any $\varepsilon > 0$, $t \to W_\varepsilon(t)$ is a non increasing function of t.

It results from Helly's Theorem (see for instance in [29]):

Corollary 6.1. *There exists $(\varepsilon_n)_{n\in\mathbb{N}}$ that goes to zero when $n \to \infty$, a non increasing function $\widetilde{W}(t)$ such that for all $t \geq 0$, $W_{\varepsilon_n}(t) \to \widetilde{W}(t)$ as $n \to \infty$.*

From now we will consider this sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Let us write the identity (6.2) under the form

$$\int_{\mathbb{R}^3} \overline{\mathbf{u}_0}(\mathbf{x}) \cdot \mathbf{w}(0, \mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{u}_{\varepsilon_n}(t, \mathbf{x}) \cdot \mathbf{w}(t, \mathbf{x}) d\mathbf{x} + I_{1,\varepsilon_n}(t, \mathbf{w}) + I_{2,\varepsilon_n}(t, \mathbf{w}). \tag{6.4}$$

We will prove in this section the following results, which will complete the proof of Theorem 2.3.

Lemma 6.2. *There exists a nondecreasing sequence $(n_j)_{j\in\mathbb{N}}$ such that for all $t \geq 0$, for all $\mathbf{w} \in E_\sigma$, the sequences $(I_{1,\varepsilon_{n_j}}(t, \mathbf{w}))_{j\in\mathbb{N}}$ and $(I_{2,\varepsilon_{n_j}}(t, \mathbf{w}))_{j\in\mathbb{N}}$ are convergent sequences.*

Lemma 6.3. *For each fixed time t, there exists $\mathbf{u}(t, \cdot) \in L^2(\mathbb{R}^3)^3$ such that $(\mathbf{u}_{\varepsilon_{n_j}}(t, \cdot))_{n\in\mathbb{N}}$ weakly converges to $\mathbf{u}$ in $L^2(\mathbb{R}^3)^3$.*

Lemma 6.4. *There exists a set $A \subset \mathbb{R}_+$ the complementary of which is a zero measure set and such that for all $t \in A$, the sequence $(\mathbf{u}_{\varepsilon_{n_j}}(t, \cdot))_{n\in\mathbb{N}}$ strongly converges to $\mathbf{u}(t, \cdot)$ in $L^2(\mathbb{R}^3)^3$.*

Lemma 6.5. *The field $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is a turbulent solution to the NSE (1.1).*

This program is divided into two subsections. The first is devoted to prove Lemma 6.2, divided in turn into two sub-subsections, one considering $(I_{1,\varepsilon_{n_j}}(t, \mathbf{w}))_{j\in\mathbb{N}}$, the other $(I_{2,\varepsilon_{n_j}}(t, \mathbf{w}))_{j\in\mathbb{N}}$. In the second subsection we prove Lemma 6.3, 6.4 and 6.5 one after another.

6.2 Weak convergence and measures

In all what follows, $\mathbf{w}$ is any given field of E_σ, $t \in \mathbb{R}_+$.

6.2.1 Convergence of $I_{1,\varepsilon_n}(t, \mathbf{w})$

As $(\mathbf{u}_{\varepsilon_n})_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)^3) = (L^1(\mathbb{R}_+, L^2(\mathbb{R}^3)^3))'$, we can extract from the sequence $(\mathbf{u}_{\varepsilon_n})_{n\in\mathbb{N}}$ a subsequence which converges to a field $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}(t, \mathbf{x}) \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$ for the weak $\star$ topology of $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$. We still denote this subsequence $(\mathbf{u}_{\varepsilon_n})_{n\in\mathbb{N}}$ for the simplicity. We show in what follows:

$$\begin{aligned} &\lim_{n\to\infty} I_{1,\varepsilon_n}(t, \mathbf{w}) = \\ &\qquad \int_0^t \int_{\mathbb{R}^3} \widetilde{\mathbf{u}}(t', \mathbf{x}) \cdot \left[\nu \Delta \mathbf{w}(t', \mathbf{x}) + \mathrm{div}(A \nabla \mathbf{w})(t', \mathbf{x}) + \frac{\partial \mathbf{w}}{\partial t'}(t', \mathbf{x}) \right] d\mathbf{x} dt'. \end{aligned} \tag{6.5}$$

From the definition of E_σ (see (6.1)), $\Delta\mathbf{w}$, $\dfrac{\partial \mathbf{w}}{\partial t} \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$, thereby

$$
\begin{aligned}
(6.6)\qquad & \lim_{n\to\infty} \int_0^t \int_{\mathbf{R}^3} \mathbf{u}_{\varepsilon_n}(t',\mathbf{x}) \cdot \left[\nu\Delta\mathbf{w}(t',\mathbf{x}) + \frac{\partial \mathbf{w}}{\partial t'}(t',\mathbf{x})\right] d\mathbf{x}dt' = \\
& \qquad \int_0^t \int_{\mathbf{R}^3} \widetilde{\mathbf{u}}(t',\mathbf{x}) \cdot \left[\nu\Delta\mathbf{w}(t',\mathbf{x}) + \frac{\partial \mathbf{w}}{\partial t'}(t',\mathbf{x})\right] d\mathbf{x}dt'.
\end{aligned}
$$

It remains to pass to the limit in the term

$$
\int_0^t \int_{\mathbf{R}^3} \mathbf{u}_{\varepsilon_n}(t',\mathbf{x}) \,\mathrm{div}(\overline{A\nabla\overline{\mathbf{w}}})(t',\mathbf{x}) d\mathbf{x}dt'.
$$

Let Δ_{ε_n} denotes the difference

$$
\Delta_{\varepsilon_n} = \int_0^t \int_{\mathbf{R}^3} \mathbf{u}_{\varepsilon_n}(t',\mathbf{x}) \,\mathrm{div}(\overline{A\nabla\overline{\mathbf{w}}})(t',\mathbf{x}) d\mathbf{x}dt' - \int_0^t \int_{\mathbf{R}^3} \widetilde{\mathbf{u}}(t',\mathbf{x}) \,\mathrm{div}(A\nabla\mathbf{w})(t',\mathbf{x}) d\mathbf{x}dt',
$$

that we split as

$$
\begin{aligned}
\Delta_{\varepsilon_n} &= \int_0^t \int_{\mathbf{R}^3} (\mathbf{u}_{\varepsilon_n}(t',\mathbf{x}) - \widetilde{\mathbf{u}}(t',\mathbf{x})) \,\mathrm{div}(A\nabla\mathbf{w})(t',\mathbf{x}) d\mathbf{x}dt' + \\
&\quad + \int_0^t \int_{\mathbf{R}^3} (\mathbf{u}_{\varepsilon_n}(t',\mathbf{x})(\mathrm{div}(\overline{A\nabla\overline{\mathbf{w}}})(t',\mathbf{x}) - \mathrm{div}(A\nabla\mathbf{w})(t',\mathbf{x})) d\mathbf{x}dt' \\
&= \Delta_{\varepsilon_n,1} + \Delta_{\varepsilon_n,2}.
\end{aligned}
$$

As $\mathbf{w} \in \mathrm{E}_\sigma$ and $A \in C_b(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3))$, then $\mathrm{div}(A\nabla\mathbf{w}) \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$, leading to $\Delta_{\varepsilon_n,1} \to 0$ as $n \to \infty$. Moreover, the Cauchy-Schwarz inequality and the energy balance yield

$$
|\Delta_{\varepsilon_n,2}| \le \sqrt{W(0)} \int_0^t ||\mathrm{div}(\overline{A\nabla\overline{\mathbf{w}}})(t',\cdot) - \mathrm{div}(A\nabla\mathbf{w})(t',\cdot))||_{0,2} dt'.
$$

By (3.3) and algebraic calculations, we get

$$
||\mathrm{div}(\overline{A\nabla\overline{\mathbf{w}}})(t',\cdot) - \mathrm{div}(A\nabla\mathbf{w})(t',\cdot))||_{0,2} \le C\varepsilon_n ||A||_{1,\infty} ||\mathbf{w}||_{3,2},
$$

hence[4] $\Delta_{\varepsilon_n,2} \to 0$ as $n \to \infty$, again because $A \in C_b(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3))$ and $\mathbf{w} \in L^1_{loc}(\mathbb{R}_+, H^3(\mathbb{R}^3)^3)$. In conclusion, we obtain

$$
\begin{aligned}
(6.7)\qquad & \lim_{n\to\infty} \int_0^t \int_{\mathbf{R}^3} \mathbf{u}_{\varepsilon_n}(t',\mathbf{x}) \,\mathrm{div}(\overline{A\nabla\overline{\mathbf{w}}})(t',\mathbf{x}) d\mathbf{x}dt' = \\
& \qquad \int_0^t \int_{\mathbf{R}^3} \widetilde{\mathbf{u}}(t',\mathbf{x}) \,\mathrm{div}(A\nabla\mathbf{w})(t',\mathbf{x}) d\mathbf{x}dt',
\end{aligned}
$$

hence (6.5). In the following, we shall denote by $I_1(t,\mathbf{w})$ the limit of $(I_{1,\varepsilon_n}(t,\mathbf{w}))_{n\in\mathbf{N}}$, where $(I_{1,\varepsilon_n}(t,\mathbf{w}))_{n\in\mathbf{N}}$ is given by (6.2) and (6.4).

[4] This is where we need $\nabla A \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. It is likely that there is a way to do without this hypothesis.

6.2.2 Convergence of $I_{2,\varepsilon_n}(t,\mathbf{w})$

We show below step by step that $I_{2,\varepsilon_n}(t,\mathbf{w})$ converges to a limit denoted by $I_2(t,\mathbf{w})$, which is not well identified at this stage. Indeed, for now we still do not know whether $(\mathbf{u}_{\varepsilon_n})_{n\in\mathbb{N}}$ is compact in any L^p space, since standard compactness results do not seem to apply in this case. Therefore, we are not able to directly take the limit in the convective term, that we treat in this section as a sequence of measures, denoted by $(\mu_n)_{n\in\mathbb{N}}$. As we are working in the whole space by using a very weak formulation, we are led to consider restrictions of the μ_n's over balls B_k centered at the origin and of radius $k\in\mathbb{N}$, in which standard compactness results apply. Then we use the valuable estimate (5.27) to select the appropriate radius k to be able to pass to the limit. This convergence analysis is divided into three steps:

a) Definition of the convective measures,

b) Compactness on balls B_k,

c) Passing to the limit.

a) The Convective measures are defined by

$$\mu_n = \overline{\mathbf{u}_{\varepsilon_n}} \otimes \mathbf{u}_{\varepsilon_n}. \tag{6.8}$$

Let us fix a given time $T>0$. We will study the sequence $(\mu_n)_{n\in\mathbb{N}}$ on $[0,T]$ for technical conveniences, which is not restrictive since T may be chosen as large as we want. The energy balance yields

$$\forall\, t'\in[0,T], \qquad \int_{\mathbb{R}^3} |\mu_n(t',\mathbf{x})| d\mathbf{x}dt' \le W(0),$$

hence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T],L^1(\mathbb{R}^3)^9)$, which suggests to treat each component of μ_n as measures.

b) Compactness. Let $k\in\mathbb{N}$, $B_k=B(\mathrm{O},k)\subset\mathbb{R}^3$ be the ball centered at the origin O of radius k. We denote by $M(B_k)$ the set of radon measures over B_k. Therefore, the considerations above show that any k being fixed,

$$\text{the sequence} \quad (\mu_n)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty([0,T],M(B_k)^9) = (L^1([0,T],C(B_k)^9))'.$$

Let μ_n^k denotes the restriction of μ_n to the ball B_k,

$$\mu_n^k = \mu_n|_{B_k}. \tag{6.9}$$

We deduce from the Banach-Aloaglu theorem combined with the Cantor diagonal argument that there exists a sequence $(n_j)_{j\in\mathbb{N}}$ such that each sequence of measures $(\mu_{n_j}^k)_{j\in\mathbb{N}}$ converges to a measure μ^k in $L^\infty([0,T],M(B_k)^9)$ weak $\star$. Moreover,

$$\forall\, k<k', \quad \mu^{k'}|_{B_k} = \mu^k.$$

c) Passing to the limit. We show the convergence of the sequence $(I_{2,\varepsilon_{n_j}}(t,\mathbf{w}))_{j\in\mathbb{N}}$ by the Cauchy criterion in using the estimate (5.27) of Lemma 5.4, which guarantees that the μ_n's have low mass distributions at infinity. Let $\mathbf{a}=\mathbf{a}(t,\mathbf{x})\in L^1([0,T],C_b(\mathbb{R}^3)^9)$, where $C_b(\mathbb{R}^3)$ denotes the space of bounded continuous functions on $\mathbb{R}^3$, $t\in[0,T]$. Let

$$\lambda_n(t,\mathbf{a}) = \int_0^t \int_{\mathbb{R}^3} \mu_n(t'\mathbf{x}) : \mathbf{a}(t',\mathbf{x})\, d\mathbf{x}dt',$$

$\eta > 0$, and $k \in \mathbb{N}$, the choice of which will be decided later. We have

$$
(6.10)\qquad
\begin{aligned}
|\lambda_{n_p}(t,\mathbf{a}) - \lambda_{n_q}(t,\mathbf{a})| \leq \int_0^t ||\mathbf{a}(t',\cdot)||_{0,\infty} \left(\int_{|\mathbf{x}|\geq k} |\mu_{n_p}(t',\mathbf{x}) - \mu_{n_q}(t',\mathbf{x})| d\mathbf{x} \right) dt' \\
+ \left| \int_0^t \int_{B_k} (\mu_{n_p}(t',\mathbf{x}) - \mu_{n_q}(t',\mathbf{x})) : \mathbf{a}(t',\mathbf{x})\, d\mathbf{x} dt' \right|.
\end{aligned}
$$

The function φ involved in (5.27) being non increasing, we get by (5.27) for $k \geq 2R_0$,

$$
(6.11)\qquad
\begin{aligned}
\int_0^t ||\mathbf{a}(t',\cdot)||_{0,\infty} \left(\int_{|\mathbf{x}|\geq k} |\mu_{n_p}(t',\mathbf{x}) - \mu_{n_q}(t',\mathbf{x})| d\mathbf{x} \right) dt' \leq \\
\left(\int_{|\mathbf{x}|\geq k/2} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \frac{4\varphi(T)}{k} \right) \int_0^T ||\mathbf{a}(t',\cdot)||_{0,\infty} dt'.
\end{aligned}
$$

As $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, we can fix $k \geq R_0$ such that the r.h.s of (6.11) is less than $\eta/2$. Furthermore, for this k and by the definition (6.9) of the μ_n^k's:

$$
\begin{aligned}
\left| \int_0^t \int_{B_k} (\mu_{n_p}(t',\mathbf{x}) - \mu_{n_q}(t,\mathbf{x})) : \mathbf{a}(t',\mathbf{x})\, d\mathbf{x} dt' \right| = \\
\left| \int_0^t \int_{B_k} (\mu^k_{n_p}(t',\mathbf{x}) - \mu^k_{n_q}(t',\mathbf{x})) : \mathbf{a}(t',\mathbf{x})\, d\mathbf{x} dt' \right| = J^k_{p,q}.
\end{aligned}
$$

From the weak convergence of the sequence $(\mu^k_{n_j})_{j\in\mathbb{N}}$, we deduce that there exists j_0 (depending on $\mathbf{a}$) such that for any $p, q \geq j_0$, we have $J^k_{p,q} \leq \eta/2$. Therefore, (6.10) gives for such p, q,

$$
|\lambda_{n_p}(t,\mathbf{a}) - \lambda_{n_q}(t,\mathbf{a})| \leq \eta.
$$

Then the sequence $(\lambda_{n_j}(t,\mathbf{a}))_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, thus convergent. Let $\lambda(t,\mathbf{a})$ denotes its limit. In view of the choice of the space E_σ, for any $t > 0$ and any $\mathbf{w} \in \mathrm{E}_\sigma$, $\nabla\mathbf{w} \in L^1([0,t], C_b(\mathbb{R}^3)^9)$. Therefore

$$
(6.12)\qquad \lim_{j\to\infty} I_{2,\varepsilon_{n_j}}(t,\mathbf{w}) = \lambda(t,\nabla\mathbf{w}).
$$

This concludes the proof of Lemma 6.2. In the following, we set $I_2(t,\mathbf{w}) = \lambda(t,\nabla\mathbf{w})$. $\square$

According to the equality (6.4), the lemma 6.2 admits the following corollary.

Corollary 6.2. *For all $\mathbf{w} \in E_\sigma$, all $t \in \mathbb{R}_+$, the sequence*

$$
\left(\int_{\mathbb{R}^3} \mathbf{u}_{\varepsilon_{n_j}}(t,\mathbf{x}) \cdot \mathbf{w}(t,\mathbf{x}) d\mathbf{x} \right)_{j\in\mathbb{N}}
$$

has a limit uniquelly determined, namely

$$
(6.13)\qquad \lim_{j\to\infty} \int_{\mathbb{R}^3} \mathbf{u}_{\varepsilon_{n_j}}(t,\mathbf{x}) \cdot \mathbf{w}(t,\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{u}_0(\mathbf{x}) \cdot \mathbf{w}(0,\mathbf{x}) d\mathbf{x} - I_1(t,\mathbf{w}) - I_2(t,\mathbf{w}).
$$

Remark 6.1. *It would not be surprising that what is done above has something to do with the Young measures (see in [3, 39, 43]). There also could be connections with the H-measures, initially introduced by L. Tartar (see [40, 41]) as well as with the work by A. Majda and R. DiPerna [15]. All of this remains to be clarified.*

6.3 Transition from weak to strong convergence: conclusion

Proof of Lemma 6.3. Consider

$$\mathbf{a} \in \bigcap_{m\geq 0} H^m(\mathbb{R}^3)^3,$$

such that $\operatorname{div} \mathbf{a} = 0$. Let $t > 0$, $\varphi = \varphi(t')$ be a non negative function of class C^∞ less than 1 such that $\varphi(t') = 1$ when $t' \in [0, t+1]$, $\varphi(t') = 0$ when $t' \in [t+2, \infty[$. It is easely checked that

$$\mathbf{w}(t', \mathbf{x}) = \varphi(t')\mathbf{a}(\mathbf{x}) \in \mathrm{E}_\sigma.$$

As $\mathbf{w}(t, \mathbf{x}) = \mathbf{a}(\mathbf{x})$, Corollary 6.2 implies that the sequence

$$\left(\int_{\mathbf{R}^3} \mathbf{u}_{\varepsilon_{n_j}}(t, \mathbf{x}) \cdot \mathbf{a}(\mathbf{x}) d\mathbf{x} \right)_{j \in \mathbf{N}}$$

has a limit uniquely determined. We recall that $||\mathbf{u}_{\varepsilon_{n_j}}(t, \cdot)||_{0,2} \leq \sqrt{W(0)}$. By Lemma 7 page 209 in [27], we can conclude that the sequence $(\mathbf{u}_{\varepsilon_{n_j}}(t, \cdot))_{j \in \mathbf{N}}$ has a weak limit in $L^2(\Omega)^2$, denoted by $\mathbf{u}(t, \cdot)$. This weak convergence also leads to:

$$\int_{\mathbf{R}^3} |\mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x} \leq \widetilde{W}(t), \tag{6.14}$$

where $\widetilde{W}$ is introduced in Corollary 6.1. □

For the simplicity, we write from now ε instead of ε_{n_j}, $\varepsilon \to 0$ instead of $j \to \infty$.

Proof of Lemma 6.4. We recall the energy balance satisfied by $\mathbf{u}_\varepsilon$

$$\frac{1}{2} W_\varepsilon(t) + \nu \int_0^t J_\varepsilon^2(t') dt' + \int_0^t K_{A,\varepsilon}^2(t') dt' = \frac{1}{2} W_\varepsilon(0) \leq \frac{1}{2} W(0), \tag{6.15}$$

where

$$\begin{cases} W_\varepsilon(t) = \displaystyle\int_{\mathbf{R}^3} |\mathbf{u}_\varepsilon(t, \mathbf{x})|^2 d\mathbf{x}, \\ J_\varepsilon^2(t) = \displaystyle\int_{\mathbf{R}^3} |\nabla \mathbf{u}_\varepsilon(t, \mathbf{x})|^2 d\mathbf{x}, \\ K_{A,\varepsilon}^2(t) = \displaystyle\int_{\mathbf{R}^3} A(t, \mathbf{x}) |\nabla \mathbf{u}_\varepsilon(t, \mathbf{x})|^2 d\mathbf{x}. \end{cases} \tag{6.16}$$

We next deduce from Fatou's Lemma and (6.15),

$$\nu \int_0^\infty (\liminf_{\varepsilon \to 0} J_\varepsilon(t'))^2 dt' \leq \frac{1}{2} W(0). \tag{6.17}$$

Therefore, $t \to \liminf_{\varepsilon \to 0} J_\varepsilon(t) \in L^1([0, \infty[)$. Then there exists a set $A \subset [0, \infty[$, such that $meas(A^c) = 0$ and

$$\forall\, t \in A, \quad \liminf_{\varepsilon \to 0} J_\varepsilon(t) < \infty. \tag{6.18}$$

Let $t \in A \cap [0, T]$ be fixed. There is a sequence $(\varepsilon_n)_{n \in \mathbf{N}}$ going to zero when $n \to \infty$ (which could depend on t) and such that the sequence $(J_{\varepsilon_n}(t))_{n \in \mathbf{N}}$ is bounded. In particular, $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbf{N}}$ is bounded in $H^1(\mathbb{R}^3)$.

Let $\eta > 0$ be given. We know from Lemma 5.4 that there exists $R > 0$ (which depends on T, $\mathbf{u}_0$ and η) and such that

$$\int_{|\mathbf{x}|\geq R} |\mathbf{u}_{\varepsilon_n}(t,\mathbf{x})|^2 d\mathbf{x} \leq \eta. \tag{6.19}$$

Furthermore, as the sequence $(\mathbf{u}_{\varepsilon_n})_{n\in\mathbf{N}}$ is bounded in $H^1(\mathbb{R}^3)$ and has a unique adherence value in $L^2(B_R)^3$ for the weak topology, the Rellich-Kondrachov theorem applies: the sequence $(\mathbf{u}_{\varepsilon_n})_{n\in\mathbf{N}}$ converges to $\mathbf{u}$ strongly in $L^2(B_R)$. Then (6.19) gives

$$\limsup_{n\to\infty} W_{\varepsilon_n}(t) \leq \int_{B_R} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} + \eta \leq \int_{\mathbf{R}^3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} + \eta, \tag{6.20}$$

which holds for any $\eta > 0$. Therefore, combining (6.20) with the definition of the function $\widetilde{W}$ and the inequality (6.14), we get:

$$\int_{\mathbf{R}^3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} \leq \widetilde{W}(t) = \limsup_{n\to\infty} W_{\varepsilon_n}(t) \leq \int_{\mathbf{R}^3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}, \tag{6.21}$$

hence the convergence of $(||\mathbf{u}_\varepsilon(t,\cdot)||_{0,2})_{\varepsilon>0}$ to $||\mathbf{u}(t,\cdot)||_{0,2}$ and the conlusion of the proof, because we already know that $(\mathbf{u}_\varepsilon(t,\cdot))_{\varepsilon>0}$ weakly converges to $\mathbf{u}(t,\cdot)$ in $L^2(\mathbb{R}^3)^3$. □

Proof of Lemma 6.5. In order to prove that $\mathbf{u}$ is a turbulent solution to the NSE, it remains to:

1) Check that $\tilde{\mathbf{u}} = \mathbf{u}$, where $\tilde{\mathbf{u}}$ was introduced in subsection 6.2 as the weak star limit of $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$,
2) Show that

$$\forall\, t \in \mathbb{R}_+, \forall\, \mathbf{w} \in \mathrm{E}_\sigma, \quad I_2(t,\mathbf{w}) = \int_0^t \int_{\mathbf{R}^3} \mathbf{u}(t',\mathbf{x}) \otimes \mathbf{u}(t',\mathbf{x}) : \nabla \mathbf{w}(t,\mathbf{x}) d\mathbf{x} dt', \tag{6.22}$$

   where $I_2(t,\mathbf{w})$ was defined by (6.12),
3) Check that $\mathbf{u}$ satifies the energy inequality,

which is done in the following each item after another.

1) *Weak star limit identification.* Let $\mathbf{a} \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^3)^3)$, and consider

$$\varphi_\varepsilon(t) = \int_{\mathbf{R}^3} \mathbf{u}_\varepsilon(t,\mathbf{x}) \cdot \mathbf{a}(t,\mathbf{x}) d\mathbf{x}.$$

According to Lemma 6.4,

$$\forall\, t \in A, \quad \lim_{\varepsilon\to 0} \varphi_\varepsilon(t) = \varphi(t) = \int_{\mathbf{R}^3} \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{a}(t,\mathbf{x}) d\mathbf{x}.$$

Moreover, the Cauchy-Schwarz inequality combined with the inequality (6.3) gives

$$|\varphi_\varepsilon(t)| \leq \sqrt{W_\varepsilon(t)}||\mathbf{a}(t,\cdot)||_{0,2} \leq \sqrt{W(0)}||\mathbf{a}(t,\cdot)||_{0,2} \in L^1(\mathbb{R}_+).$$

Because $meas(A^C) = 0$, we deduce from Lebesgue Theorem that

$$\lim_{\varepsilon\to 0} \int_0^\infty \varphi_\varepsilon(t) dt = \int_0^\infty \varphi(t) dt,$$

hence $\widetilde{\mathbf{u}} = \mathbf{u}$.

2) *Limit in the convective term.* Let $\mathbf{w} \in \mathrm{E}_\sigma$, and

$$\psi_\varepsilon(t) = \int_{\mathbf{R}^3} \overline{\mathbf{u}_\varepsilon}(t,\mathbf{x}) \otimes \mathbf{u}_\varepsilon(t,\mathbf{x}) : \nabla \mathbf{w}(t,\mathbf{x}) d\mathbf{x}.$$

Let $t \in A$. Obviously $\overline{\mathbf{u}_\varepsilon}(t,\cdot) \to \mathbf{u}(t,\cdot)$ strongly in $L^2(\mathbb{R}^3)^3$. As $\nabla \mathbf{w}$ is bounded in space and time, we deduce that for all $t \in A$,

$$\psi_\varepsilon(t) \to \psi(t) = \int_{\mathbf{R}^3} \mathbf{u}(t,\mathbf{x}) \otimes \mathbf{u}(t,\mathbf{x}) : \nabla \mathbf{w}(t,\mathbf{x}) d\mathbf{x} \quad \text{as} \quad \varepsilon \to 0,$$

and by the energy balance

$$|\psi_\varepsilon(t)| \le C W(0) ||\nabla \mathbf{w}(t,\cdot)||_\infty \in L^1([0,\infty[).$$

Then, as $meas(A^c) = 0$, we have by Lebesgue's Theorem,

$$\forall\, t \in \mathbb{R}_+, \quad \lim_{\varepsilon \to 0} \int_0^t \psi_\varepsilon(t')dt' = \int_0^t \psi(t')dt',$$

hence (6.22)

3) *Energy inequality.* It is easily checked that $\forall\, t \in A$, $\mathbf{u}_\varepsilon(t,\cdot) \to \mathbf{u}(t,\cdot)$ weakly in $H^1(\mathbb{R}^3)^3$. Therefore,

$$\left(\int_{\mathbf{R}^3} |\nabla \mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} = J(t) \le \liminf_{\varepsilon \to 0} J_\varepsilon(t),$$

and as $A \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and is non negative, a convexity argument yields

$$K_A(t) \le \liminf_{\varepsilon \to 0} K_{A,\varepsilon}(t).$$

All these inequalities hold for almost all $t \in A$. Then, by taking the limit in the energy balance (6.15) we get by (6.14) and Fatou's Lemma,

$$\frac{1}{2} W(t) + \nu \int_0^t J^2(t')dt' + \int_0^t K_A(t')dt' \le \frac{1}{2} W(0), \tag{6.23}$$

as expected, which finishes the proof. □

7 Additional remarks and open questions

7.1 Case $A = 0$ and obstruction to generalizations

Let us recall one main Leray's argument when $A = 0$, written in our framework. In this case, any regular solution to the NSE (1.1) over the time interval $[0, T[$ satisfies, according to (3.15),

$$V(t) \le V(0) + C \int_0^t \frac{V^2(t')}{\sqrt{\nu(t-t')}} dt'. \tag{7.1}$$

where $V(t)$ is defined by (2.4). The function $g(t) = 2V(0)$ is a supersolution to the non linear Volterra equation,

$$f(t) \le V(0) + C \int_0^t \frac{f^2(t')}{\sqrt{\nu(t-t')}} dt',$$

over the time interval $I = [0, 4\nu V^{-2}(0) C^{-1}]$. Therefore, by the V-maximum principle[5]

$$\forall t \in I, \quad V(t) \leq 2V(0).$$

From this, it is possible to control all the norms of any regular solutions. This is why Leray was able to construct a regular solution to the NSE (1.1) over $[0, T]$ where $T = O(\nu V^{-2}(0))$, by a fixed point process in a space equipped with the uniform norm in space. He also reported that when a singularity occurs at time T, then when $t \to T$ (see section 19 in [27]),

$$V(t) \geq C\sqrt{\frac{\nu}{T-t}}.$$

The fact that the regular solution can be extended to $[0, \infty[$ when $\nu^{-3} W(0) V(0)$ is small enough (see Theorem 2.1) is a tricky combination of such arguments, generalized as much as possible, as well as the fact that the turbulent solution becomes regular up to a time $O(W(0)^2/\nu^5)$.

When $A \neq 0$, this does not work anymore, since we get, because of the eddy viscosity term,

$$V(t) \leq V(0) + C \int_0^t \frac{V^2(t') + ||N_A||_\infty V_1(t')}{\sqrt{\nu(t-t')}} dt'. \tag{7.2}$$

where $V_1(t') = ||\nabla \mathbf{u}(t', \cdot)||_{0,\infty}$. Thereby, to control $V(t)$, we must control $V_1(t)$, which involves $V_2(t)$ and so on, and we do not know how to close this sequence of inequalities. This is why Leray's program cannot be recycled turnkey. Ideally, Oseen's work should be rewritten for the generalized Stokes problem:

$$\left\{ \begin{array}{l} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} - \operatorname{div}(A \nabla \mathbf{u}) + \nabla p = f, \\ \operatorname{div} \mathbf{u} = 0, \end{array} \right. \tag{7.3}$$

which is not done already so far we know.

In the case of the approximated system (3.5) when $A = 0$, inequality (7.1) becomes

$$V(t) \leq V(0) + C\varepsilon^{-3/2} \sqrt{W(0)} \int_0^t \frac{V(t')}{\sqrt{\nu(t-t')}} dt'. \tag{7.4}$$

Hence, by the V-maximum principle,

$$V(t) \leq f(t),$$

where $f(t)$ is the unique continuous solution to the linear Volterra equation defined over $[0, \infty[$,

$$f(t) = V(0) + C\varepsilon^{-3/2} \sqrt{W(0)} \int_0^t \frac{f(t')}{\sqrt{\nu(t-t')}} dt'.$$

From there, the analysis developed by J. Leray to estimate the time of existence of a strong solution for the NSE applies, and in this case yields the existence of a unique strong solutions to the approximated system (3.5) global in time, which does not work anymore when $A \neq 0$. This is why, everything was to be reconsidered from the beginning.

[5]To be stringent, we should discuss as in Lemma 4.1's proof to be in the framework for the application of the V-maximum principle. At this stage this is not essential and we skip here the details.

7.2 Leray-α, Bardina and others

The idea of regularizing the convection term by $\overline{\mathbf{u}} \cdot \nabla \mathbf{u}$ led to the actual concept of Leray-α model, and many other close models (NS-α, LANS-α, Clark-α, NS-Voigt, Bardina...), considered as Large Eddy Simulation models (LES) for simulating turbulent flows, although LES emerged in the 60's with Smagorinsky's work [37].
Surprisingly, Leray has planted a seed that germinates these last two decades in the field of modern LES. See for instance in Ali [1, 2], Berselli-Iliescu-Layton [6], Foias-Holm-Titi [16], Gibbon-Holm [20, 21], Geurt-Holm [19], llyin-Lunasin-Titi [22], Layton-Rebholz [26], Rebholz [36], this list being non exhaustive.
These models are based on a regularization calculated by the Helmholz filter determined by:

$$
-\alpha^2 \Delta \overline{\psi} + \overline{\psi} = \psi \quad \text{in } \mathbb{R}^3, \tag{7.5}
$$

where in this framework the regularizing parameter is named α instead of ε. The models are usually considered with periodic boundary conditions, more rarely in a bounded domain with the no-slip condition. The case of an unbounded domain and/or the full space was not considered before so far we know.
In Berselli-Lewandowski [5], we have investigated the simplified Bardina model in the whole space. Initially introduced by Bardina-Ferziger- Reynolds [4] for weather forecast, this model was studied in [10, 24, 25] in the case of periodic boundary conditions. In its simplified version, this model is given by the system

$$
\begin{cases} \partial_t \mathbf{u} + \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) - \nu \Delta \mathbf{u} + \nabla p = 0 & \text{in } \mathbb{R}^3, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{u}_{t=0} = \overline{\mathbf{u}_0}, \end{cases} \tag{7.6}
$$

in which the bar operator is specified by the Helmholz filter (7.5). We prove in [5] the existence of a unique regular solution to (7.6), global in time, that converges to a turbulent solution to the NSE. Attention must be paided with the initial data and the meaning of "regular solution", since the regularizing effect of the Helmholz filter is lower than that given by a molifier.
What is done in [5] is in the same spirit as what is done in the present paper, inspired by Leray'work. It remains to proceed to the same analysis for the other LES models of this α-class mentioned above.

7.3 Towards the NS-TKE model

The result of Theorem 2.3 still holds when $A \in L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^3))$. Indeed, we can approach A by a sequence $(A_\varepsilon)_{\varepsilon>0}$, where $A_\varepsilon \in C_b(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3))$, and then pass to the limit in the formulation (6.2) when $\varepsilon \to 0$. We also can consider the NSE (1.1) with a source term $\mathbf{f}$ that satisfies a suitable decay condition at infinity. However, we lose the benefit provided by the monotocity of the function $t \to W(t)$ and we must find out what is the right function to be considered to replace $t \to \widetilde{W}(t)$ introduced in Corollary 6.1. It is not clear that the best choice is $t \to \limsup_{\varepsilon \to 0} W(t)$. This point remains to be discussed, though it is not intractable. We have not considered these issues to avoid making the text more cumbersome.

However, this is the right track to tackle the problem of the NS-TKE model in the whole space:

$$\left\{\begin{array}{rcl}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathrm{div}\left[(2\nu + C_u\ell|k|^{1/2})\,\nabla\mathbf{u}\right] + \nabla p &=& 0,\\ \mathrm{div}\,\mathbf{u} &=& 0,\\ \partial_t k + \mathbf{u}\cdot\nabla k - \mathrm{div}((\mu + C_k\ell|k|^{1/2})\nabla k) &=& C_u\ell|k|^{1/2}|\nabla\mathbf{u}|^2 - \ell^{-1}k|k|^{1/2},\\ (\mathbf{u}, k)_{t=0} &=& (\mathbf{u}_0, k_0),\end{array}\right.$$

which is the basic "Reynolds Averaged Navier Stokes" model of turbulence (see in [11]). In this system, $(\mathbf{u}, p)$ is the mean flow field, and k the turbulent kinetic energy, that measures the intensity of the turbulence in a turbulent flow. The function $(t, \mathbf{x}) \to \ell(t, \mathbf{x})$ is the Prandlt mixing lenght, at this stage a given non negative function, C_u C_k are experimental constants. This problem was initially studied in [28] in a bounded domain $\Omega \subset \mathbb{R}^3$, with homogeneous boundary conditions. The case of $\mathbb{R}^3$ yields a very hard mathematical problem.

Appendices

A Estimates for the Oseen tensor

This appendice has entirely been written by Paul Alphonse and Adrien Laurent.

Theorem A.1. *For $t \in \mathbb{R}^{+*}$ and $|x| > 0$, let*

$$G(t, x) = \frac{1}{|x|}\int_0^{|x|} \frac{e^{-\frac{\rho^2}{4\nu t}}}{\sqrt{t}}\mathrm{d}\rho.$$

Let

$$T_{ii} = -\frac{\partial^2 G}{\partial x_j^2} - \frac{\partial^2 G}{\partial x_k^2} \text{ and } T_{ij} = \frac{\partial^2 G}{\partial x_i \partial x_j}$$

be the Oseen tensor. Then the followig estimates are verified :

$$|T(t, x)| \leq \frac{C}{(|x|^2 + \nu t)^{\frac{3}{2}}},$$

$$|D^m T(t, x)| \leq \frac{C_m}{(|x|^2 + \nu t)^{\frac{m+3}{2}}}.$$

Proof. The function G can be extended on $|x| = 0$ as a $\mathcal{C}^\infty$ function. We have

$$\frac{\partial G}{\partial x_i}(t, x) = \frac{x_i}{|x|^2}\left(\frac{e^{-\frac{|x|^2}{4\nu t}}}{\sqrt{t}} - G(t, x)\right).$$

Integrating by part G yields

$$G(t, x) = \frac{e^{-\frac{|x|^2}{4\nu t}}}{\sqrt{t}} + \frac{1}{2\nu t^{\frac{3}{2}}\,|x|}\int_0^{|x|} \rho^2\frac{e^{-\frac{\rho^2}{4\nu t}}}{\sqrt{t}}\mathrm{d}\rho.$$

Thus

$$\frac{\partial G}{\partial x_i}(t, x) = -\frac{x_i}{2\nu t^{\frac{3}{2}}\,|x|^3}\int_0^{|x|} \rho^2\frac{e^{-\frac{\rho^2}{4\nu t}}}{\sqrt{t}}\mathrm{d}\rho.$$

With this same trick, one finds

$$\frac{\partial^2 G}{\partial x_i \partial x_j}(t,x) = -\frac{1}{6\nu t^{\frac{3}{2}}} e^{-\frac{|x|^2}{4\nu t}} \delta_{ij} + \left(\frac{x_i x_j}{4\nu^2 t^{\frac{5}{2}} |x|^5} - \frac{1}{12\nu^2 t^{\frac{5}{2}} |x|^3} \delta_{ij} \right) \int_0^{|x|} \rho^4 e^{-\frac{\rho^2}{4\nu t}} \mathrm{d}\rho.$$

And by a change of variables,

$$\frac{\partial^2 G}{\partial x_i \partial x_j}(t,x) = -\frac{1}{6\nu t^{\frac{3}{2}}} e^{-\frac{|x|^2}{4\nu t}} \delta_{ij} + \left(\frac{8\nu^{\frac{1}{2}} x_i x_j}{|x|^5} - \frac{8\nu^{\frac{1}{2}}}{3|x|^3} \delta_{ij} \right) \int_0^{\frac{|x|}{2\sqrt{\nu t}}} \rho^4 e^{-\rho^2} \mathrm{d}\rho.$$

We then have

$$|T_{ij}| \lesssim \frac{1}{t^{\frac{3}{2}}} e^{-\frac{|x|^2}{4\nu t}} + \frac{1}{|x|^3} \int_0^{\frac{|x|}{2\sqrt{\nu t}}} \rho^4 e^{-\rho^2} \mathrm{d}\rho.$$

Finally, by denoting $y = \frac{|x|}{2\sqrt{\nu t}}$, we have

$$(|x|^2 + \nu t)^{\frac{3}{2}} |T_{ij}| \lesssim (1+y^2)^{\frac{3}{2}} e^{-y^2} + (1 + \frac{1}{y^2})^{\frac{3}{2}} \int_0^y \rho^4 e^{-\rho^2} \mathrm{d}\rho.$$

The first term of the inequality is bounded for all $y \in \mathbb{R}^+$. We then denote $\varphi(y)$ the function corresponding to the second term of the inequality. The function φ is continuous on $\mathbb{R}^{+*}$ and verifies

$$\lim_{y \to +\infty} \varphi(y) = \int_0^{+\infty} \rho^4 e^{-\rho^2} \mathrm{d}\rho < +\infty.$$

For the case $y \to 0$, we notice by integrals comparison that

$$\int_0^y \rho^4 e^{-\rho^2} \mathrm{d}\rho \underset{y \to 0}{\sim} \frac{y^5}{5}.$$

Then

$$\varphi(y) \underset{y \to 0}{\sim} \frac{y^2}{5},$$

and φ is bounded on $\mathbb{R}^+$. This work gives us that

$$|T(t,x)| \leq \frac{C}{(|x|^2 + \nu t)^{\frac{3}{2}}}.$$

For the estimates on the derivatives, one can show by induction that

$$\left| \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} \frac{\partial^2 G}{\partial x_i \partial x_j} \right| (t,x) \lesssim P_m(y) \frac{1}{t^{\frac{m+3}{2}}} e^{-y^2} + \frac{1}{|x|^{m+3}} \int_0^y \rho^{4+2m} e^{-\rho^2} \mathrm{d}\rho,$$

where P_m is a polynomial of degree m. Adapting the same method as before yields the estimate on $D^m T$. $\square$

B Non linear Volterra equations and V-maximum principle

The results of this section about the non linear Volterra equations and the V-maximum principle are due to Luc Tartar.

B.1 Framework

Let $a \in \mathbb{R}_+$, $T \in \mathbb{R}_+^\star$, $K \in L^1([0,T])$, $k \geq 0$ a.e. in $[0,T]$, P a continuous non increasing real valued function. We consider the following functional equation,

$$f(t) = a + \int_0^t K(t-t')P(f(t'))dt'. \tag{B.1}$$

When $P(f) = f$, this equation is a Volterra equation. As we have seen in this paper, we have to consider P that are not linear, and this is why we call this equation a generalized non linear Volterra equation. The aim of this appendix is to prove a maximum principle which states that subsolutions of (B.1) are below supersolutions.
In this section, we define the notions of sub and super solutions, and we show how to construct solutions from these sub-super solutions. In the following, we set for any $f \in L^\infty([0,T])$, $a \geq 0$,

$$S[a,f](t) = a + \int_0^t K(t-t')P(f(t'))dt', \quad t \in [0,T]. \tag{B.2}$$

We first note that as P is non increasing and $K \geq 0$, when $f \leq g$, then $S[a,f] \leq S[a,g]$. Moreover, when $f \in L^\infty([0,T]$, then $S[a,f] \in C([0,T])$.

Definition B.1. *We say that $f \in L^\infty([0,T])$ is a subsolution of (B.1) if $f \leq S[a,f]$. We say that $g \in L^\infty([0,T])$ is a supersolution of (B.1) if $S[a,g] \leq g$.*

Remark B.1. *We remark that $f = 0$ is always a subsolution of (B.1). However,it is important to note that the solution of (B.1) may be not defined over $[0,T]$. Take for instance $K = 1$, $P(z) = z^2$. Then (B.1) becomes the differential equation $f' = f^2$, $f(0) = 0$, whose solution is $f(t) = a(1-at)^{-1}$, which blows up at a time less than T when $aT > 1$. In such case, there is no supersolution over $[0,T]$.*

As a consequence of the assumption $K \in L^1([0,T])$, the following result is straightforward.

Lemma B.1. *Let $G > a$. Then there exists $\tau \in]0,T]$ such that $g(t) = G$ is a supersolution of (B.1) over $[0,\tau]$.*

Assume now that there exists a supersolution $g \geq 0$ of (B.1) over $[0,T]$, and let $(g_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$g_0 = g, \quad g_{n+1} = S[a,g_n].$$

We obvioulsy have $0 \leq g_{n+1} \leq g_n$ for all n, and

Lemma B.2. *The sequence $(g_n)_{n\in\mathbb{N}}$ uniformly converges to a solution of (B.1).*

Proof. We first observe that g_n is continuous for $n \geq 1$. By monotonicity and since $g_n \geq 0$, $(g_n)_{n\in\mathbb{N}}$ simply converges to some $f^+ \in L^\infty([0,T])$. As for $n \geq 1$

$$|K(t-t')P(g_n(t'))| \leq K(t-t')\max(|P(0)|, |P(\max_{[0,T]} g_1)|) \in L^1([0,t]),$$

and P is continous, we deduce from Lebesgue's Theorem that for all $t \in [0,T]$, $S[a,g_n](t)$ converges to $S[a,f^+](t)$. The inequalities $g_{n+1} \leq g_n$ yields that f^+ is a solution of (B.1), hence f^+ is continuous and by Dini's Theorem, the convergence of the sequence $(g_n)_{n\in\mathbb{N}}$ is uniform. We also notice that $f^+ \leq g_n$ for all n. $\square$

B.2 Uniqueness

The solution to (B.1) may be not unique. For instance, take $K = 1$ and $P(z) = \sqrt{z}$, $a = 0$. Therefore (B.1) becomes the differential equation $f' = \sqrt{f}$, $f(0) = 0$, whose solutions are $f(t) = 0$ and $f(t) = t^2/4$. However, when P is Lipchitz, uniqueness occurs in some sense, which is the aim of this section.
Assume that (B.1) has a subsolution f and a supersolution g that verify $f \le g$, both being continuous. Arguing as in lemma B.2, we see in this case that the sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_0 = f$, $f_{n+1} = S[a, f_n]$, uniformly converges to a solution f^- of (B.1), that also satisfies $f^- \le f^+$.

Remark B.2. *As* 0 *is a subsolution, according to Lemma B.2, we have shown that there exists* $\tau > 0$ *such that the nonlinear Volterra equation (B.1) has a solution over* $[0, \tau]$

Our uniqueness result is phrased as follows.

Lemma B.3. *Assume that* P *is a non increasing Lipschitz continuous function,* $K \in L^1([0,T])$*. Then* $f^+ = f^-$ *over* $[0,T]$.

Proof. Let L denotes the Lipchitz constant of P. Then we have

$$\forall t \in [0,T], \quad 0 \le f^+(t) - f^-(t) \le L \int_0^t K(t-t')(f^+(t') - f^-(t'))dt'. \tag{B.3}$$

We first assume that K is bounded by a constant M. Therefore, (B.3) yields

$$\forall t \in [0,T], \quad 0 \le f^+(t) - f^-(t) \le LM \int_0^t (f^+(t') - f^-(t'))dt', \tag{B.4}$$

from which we easely deduce

$$\forall t \in [0,T], \forall m \ge 2, \quad 0 \le f^+(t) - f^-(t) \le \frac{(LMt)^m}{m!} \sup_{t'\in[0,T]} (f^+(t') - f^-(t')), \tag{B.5}$$

hence $f^+ = f^-$. For $K \in L^1([0,T])$, we rephrase (B.3) as

$$0 \le \varepsilon \le \Phi(\varepsilon),$$

by writting $\varepsilon = f^+ - f^-$, and

$$\Phi(u)(t) = L \int_0^t K(t-t')u(t')dt'.$$

The operator Φ is a linear operator, the kernel of which is equal to $\widetilde{K}(t) = K(t)1\!\!1_{[0,t]}$. The kernel of the operator Φ^2 is equal to $\widetilde{K} \star \widetilde{K}$, which is continuous, then bounded on the compact $[0,T]$, which yields a similar inequality as (B.5) and the conclusion $f^+ = f^-$. $\square$

Remark B.3. *With the assumptions of Lemma B.3, When* P *is linear, that is* $P(f) = \alpha_1 + \alpha_2 f$ $(\alpha_i \ge 0)$*, it easy checked that the solution of (B.1) can be extended over* $[0, \infty[$. *In this case, (B.1) is referred to as linear Volterra equation.*

B.3 V-maximum principle

The aim of this section is to prove the following result. We still assume $K \in L^1([0,T])$ and that P is a non increasing Lipchitz-continuous function.

Lemma B.4. *Let f be a subsolution of (B.1) and g a supersolution of (B.1) over $[0,T]$. Then*

$$\forall\, t \in [0,T], \quad f(t) \leq g(t). \tag{B.6}$$

Proof. By considering $S[a,f]$ instead of f, we can assume that f is continuous without loss of generality. Similarly, by considering the sequence $(g_n)_{n\in\mathbb{N}}$ as in Lemma B.2, we can assume that g is a solution of (B.1) instead being a supersolution.
Assume that (B.6) do not hold, and let

$$\tau = \sup\{t \in [0,T[,\text{ s.t.} \,\forall\, t' \in [0,t],\ f(t') \leq g(t')\} \tag{B.7}$$

Our assumption yiels $\tau < T$ and there exists a sequence $(t_n)_{n\in N}$ that converges to τ, such that $t_n > \tau$ for each n and $f(t_n) > g(t_n)$. We may have $\tau = 0$.
Given $\eta > 0$, let $k = k(t)$ be the function defined by

$$k(t) = \left\{ \begin{array}{ll} g(t), & t \in [0,\tau], \\ g(t) + \eta, & t \in]\tau, T]. \end{array} \right. \tag{B.8}$$

We claim that there exists $S > \tau$ such that k is a supersolution of (B.1) over $[0,S]$ and $f \leq k$ over $[0,S]$. Indeed, as P is Lipchitz continuous and non increasing,

$$\forall\, t' \in [\tau, T], \quad 0 \leq P(k(t')) - P(g(t')) \leq L\eta. \tag{B.9}$$

Therefore, since g is a solution of (B.1), k is a supersolution of (B.1) over $[0,S]$ for all $S > \tau$ that satisfy

$$\forall\, t \in [\tau, S], \quad L\eta \int_\tau^t K(t-t')dt' \leq \eta. \tag{B.10}$$

As $K \in L^1([0,T])$ there exists $S_0 > \tau$ such that for all $S \in]\tau, S_0]$, (B.10) holds. Furthermore, as f is continuous, there exists $S \in]\tau, S_0]$ such that

$$\forall\, t \in [\tau, S], \quad f(t) \leq k(t).$$

We consider the sequences, over $[0,S]$

$$f_0 = f, \quad f_{n+1} = S[a, f_n], \quad k_0 = k, \quad k_{n+1} = S[a, k_n].$$

we have over $[0,S]$ and for all n,

$$f \leq f_n \leq k_n \leq k.$$

According to Lemma B.2, $(k_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ converge to a solution of (B.1) over $[0,S]$ which is above f over $[0,T]$. By the uniqueness result of Lemma B.3, this solution is the restriction of h to $[0,S]$, which contradicts the definition of τ and concludes the proof. $\square$

8 Compliance with Ethical Standards

The author is member of the "Institut de Recherche Mathematique de Rennes" of the University of Rennes 1. He is employed by the University of Rennes 1. For this reason, he is a french official who do not perceive any additional salary other than that provided by the university of Rennes 1. In particular, he did not received any additional grant for this reasearch work, which is an original work, so far he knows. Therefore, the author declare that he has no conflict of interest.

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